Exercise set 2 Algorithms and Complexity 2023 **Due 16-Oct-2023**

You may collaborate and submit answers in groups of at most two.
Good solutions are complete and concise.
Please e-mail to bouman@ese.eur.nl with subject "LNMB-AC Exercise Set 2".
Groups submissions should put the other group member in the submission e-mail's CC.

3. An instance of the MAX BISECTION problem consists of an undirected graph with vertex set $V = \{1, 2, \ldots, 2n\}$ and positive real edge weights $w(i,j)$ for $\{i,j\} \in E$. The goal is to partition V into two parts V_1 and V_2 of size n, so that the total weight of the edges between V_1 and V_2 is as large as possible. Consider the following ILP.

$$\begin{array}{lll} \max & \displaystyle\sum_{\{i,j\}\in E} w(i,j)\, z_{i,j} & \\ s.t. & z_{i,j} \;\leq\; x_i + x_j & \text{for } \{i,j\} \in E \\ & z_{i,j} \;\leq\; 2 - x_i - x_j & \text{for } \{i,j\} \in E \\ & \sum_{i=1}^{2n} x_i \;=\; n & \\ & x_i \in \{0,1\} & \text{for } i \in V \\ & z_{i,j} \in \{0,1\} & \text{for } \{i,j\} \in E \end{array}$$

(a) Show that this ILP correctly models the MAX BISECTION problem.

(b) Show that any optimal solution x and z of the ILP satisfies $z_{ij} = x_i + x_j - 2x_i x_j$.

Next, let us consider the LP relaxation of this ILP, where the integrality constraints $x_i \in \{0,1\}$ and $z_{i,j} \in \{0,1\}$ are relaxed to the continuous constraints $0 \leq x_i \leq 1$ and $0 \leq z_{i,j} \leq 1$. Furthermore we define the auxiliary value $F(x) := \sum_{\{i,j\}\in E} w(i,j)\,(x_i + x_j - 2x_i x_j)$.

(c) Prove that any feasible solution x and z of the LP relaxation satisfies the following inequality: [1]

$$F(x) \;\geq\; \frac{1}{2} \sum_{\{i,j\}\in E} w(i,j)\, z_{i,j}$$

(d) Consider a feasible solution for the LP with the property that for each edge $\{i,j\} \in E$ either $x_i \geq \frac{1}{2} \geq x_j$ or $x_j \geq \frac{1}{2} \geq x_i$ which has at least two fractional variables, x_i and x_j. Argue that it is possible to increase one variable by $\epsilon > 0$ and to decrease the other one by the same ϵ, such that the solution stays feasible for the LP, the value of $F(x)$ does not decrease and one of the two variables becomes integer.

(e) Use these arguments to design a polynomial time approximation algorithm for MAX BISECTION that yields at least 1/2 of the optimal objective value.

4. An instance of the MIN-RADIUS-FACILITY-LOCATION (MRFL) problem consists of an integer k, cities $\{1, \ldots, n\}$, and integer distances $d(i,j)$ for every pair of cities i, j. The distances have the following properties for all cities x, y, z:

$$(a)\; d(x,x) = 0, \qquad (b)\; d(x,y) = d(y,x), \qquad (c)\; d(x,y) + d(y,z) \geq d(x,z).$$

If $C \subseteq \{1, \ldots, n\}$ is a set of cities and j is a city, we denote $dist(C,j) := \min\{d(i,j) : i \in C\}$ for the minimum distance from j to a city in C (so $dist(C,j) = 0$ if $j \in C$). The goal in the MRFL problem is to find a set C of size at most k minimizing the radius $r(C) := \max_{1 \leq j \leq n} dist(C,j)$.

(a) Give a polynomial time algorithm that takes an instance of MRFL and a number t as input, and finds a set C of size at most k satisfying $r(C) \leq 2t$ if there exists a set C^* of size at most k with $r(C^*) \leq t$.

(b) Use the algorithm from (a) to give a 2-approximation for MRFL. Your algorithm may run in pseudo-polynomial time.

(c) ***Bonus***[2]***:*** Give a polynomial time 2-approximation for MRFL.

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[1] Hint: Use the AM-GM inequality (stating that, for non-negative a, b, $4ab \leq (a+b)^2$) to show that any feasible solution of the LP satisfies $\frac{1}{2} z_{i,j} \leq x_i + x_j - 2x_i x_j$ for every i and j.

[2] You do not need to answer this question to get full points, but a fully correct answer to this question will give you full points for Question 4 plus a bonus point even when (a) and (b) are skipped.