Algorithms and Complexity (AC)

Third Lecture (25th of September, 2023)

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September – November 2023

LNMB – Landelijk Netwerk Mathematische Besliskunde

Table of contents

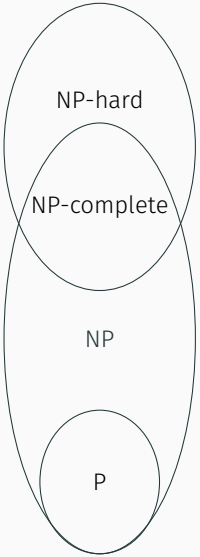
Our program for week 3

- Questions so far or about the assignment?
- pseudo-polynomial time, strong NP-hardness & weak NP-hardness
- co-NP, co-NP versus NP
- Beyond P, NP and coNP
- Exercises 1: questions?, partner-finding?

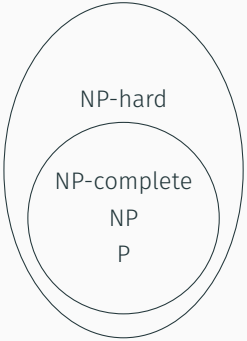
More NP-hardness

Complexity Classes

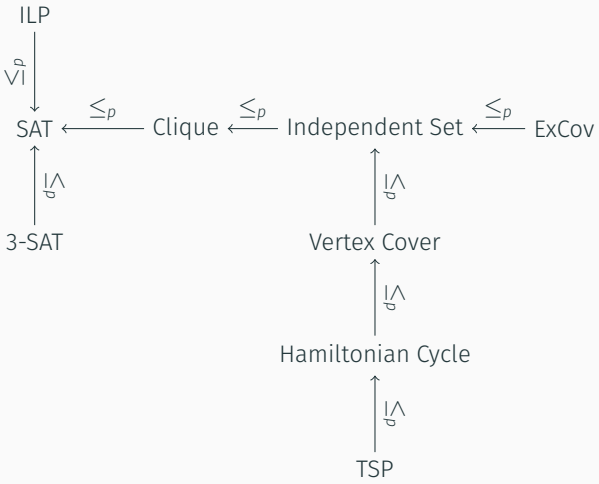
If $P \neq NP$



If $P = NP$



Reductions


ILP
$\leq_p$
SAT
$\leq_p$
Clique
$\leq_p$
Independent Set
$\leq_p$
ExCov
$\geq_p$
3-SAT
$\geq_p$
Vertex Cover
$\geq_p$
Hamiltonian Cycle
$\geq_p$
TSP

Subset Sum (SS)

Instance: positive integers $a_1, \ldots, a_n$; a bound b

Question: does there exist an index set $J \subseteq \{1, \ldots, n\}$ with $\sum_{j \in J} a_j = b$?

Example: $(a_1, \ldots, a_{12}) = (1, \ldots, 12), b = 50$. Yes or no instance?

Yes: $1 + 2 + 3 + 4 + 6 + 7 + 8 + 9 + 10 = 50$.

We argued last week that this problem is in NP.

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We argued last week that this problem is in NP. What about NP-completeness/NP-hardness?

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Let us try a reduction from Exact Cover.

Exact cover (Ex-Cov)

Instance: a ground set X; subsets $S_1, \ldots, S_m$ of X

Question: do there exist some subsets S_i that form a partition of X?

Subset Sum

First idea: encode the sets at binary strings and interpret these as numbers

Set	Binary Encoding				Decimal Value
$\{1, 3\}$	0	1	0	1	5
$\{2, 4\}$	1	0	1	0	10
$\{2, 3, 4\}$	1	1	1	0	14
$\{4\}$	1	0	0	0	8

Any objections?

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Carry-over is a problem:

$$\begin{array}{cccc} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ \hline 1 & 1 & 1 & 0 \end{array} +$$

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Any suggestion how to resolve this?

Second idea: create enough space in the encoding so carry-over can not reach the next element

Set	Binary Encoding				Decimal Value
$\{1, 3\}$	000	001	000	001	65
$\{2, 4\}$	001	000	001	000	520
$\{2, 3, 4\}$	001	001	001	000	584
$\{4\}$	001	000	000	000	512

Now the elements can not interfere with each other:

$$\begin{array}{cccc} 000 & 001 & 000 & 001 \\ 001 & 000 & 000 & 001 \quad + \\ \hline 001 & 001 & 000 & 010 \end{array}$$

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Transformation

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Define numbers a_j as $a_j := \sum_{i=1}^{m} c_{ij} \cdot d_i$ with $c_{ij} = 1$ if $x_i \in S_j$ and $d_i = (n+1)^{i-1}$.
Set $b := \sum_{i=1}^{m} (n+1)^{i-1}$.

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2-Partition

2-PARTITION

Instance: positive integers $a_1, \ldots, a_n$ with $\sum_{i=1}^{n} a_i = 2A$.

Question: does there exist an index set $I \subseteq \{1, \ldots, n\}$ with $\sum_{i \in I} a_i = A$?

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Define $T = \sum_{i=1}^{n} a_i$. Now construct an instance of 2-Partition with $n + 2$ numbers a'_i such that $a'_i = a_i$ for $1, \ldots, n$. Then define $a'_{n+1} = T + b$ and $a'_{n+2} = 2T - b$.

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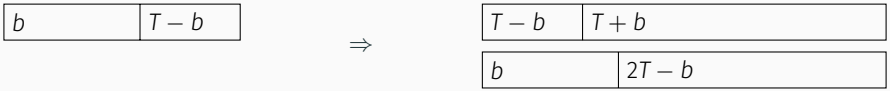

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(SS) $\Rightarrow$ **(2-Partition)**: Given a subset $\mathcal{S}$ that sums to b, determine is $T - b \geq b$, and allocate numbers a'_n and a_{n+1} according either to S or it's complement depending on this.

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(2-Partition) ⇒ (SS): Given a valid partition into two sets, consider which set contains the $a'_{n+2} = 2T - b$ element. As the total sum is $4T$, both partitions must sum up to $2T$ and neither can contain both a'_{n+1} and a'_{n+2} as that would sum up to $3T$.

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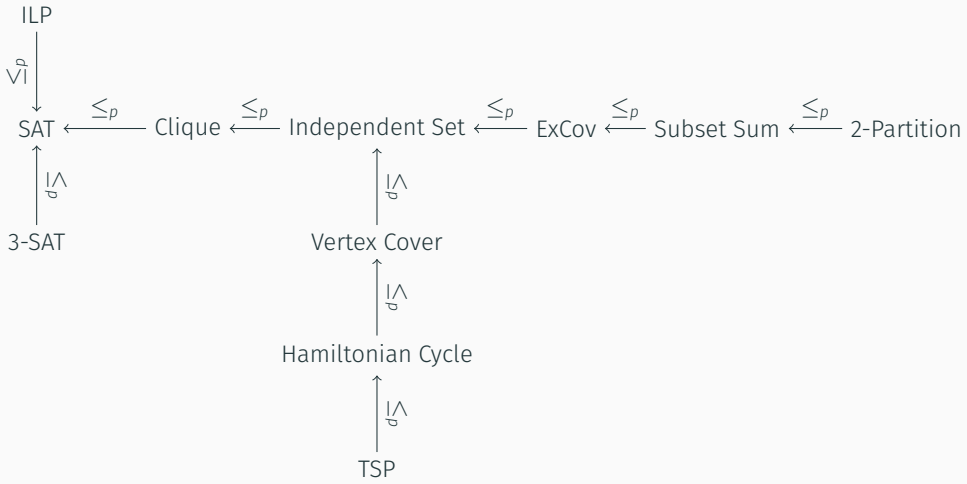
Subset Sum and 2-Partition

Theorems

Subset Sum is NP-hard and NP-complete.

2-Partition is NP-hard and NP-complete.

Reductions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Questions?

Pseudo-polynomial time

An algorithm for SUBSET SUM

Define function $F(k, c)$:
$F(k, c)$=TRUE if and only if $\exists S \subseteq \{1, \ldots, k\} : \sum_{i \in S} a_i = c$

This can be

$$F(k, c) = \begin{cases} \text{FALSE} & \text{if } k < 0 \vee c < 0 \\ \text{TRUE} & \text{if } c = 0 \\ F(k-1, c) \vee F(k-1, c-a_i) & \text{otherwise} \end{cases}$$

We can convert this recursion to a Dynamic Programming algorithm

An algorithm for SUBSET SUM

Dynamic programming algorithm to compute $F(n, b)$

```
def dp_subset_sum(numbers, target):
    k = len(numbers)
    # Initialize a Dynamic Programming Table and values F(0,j)
    table = [[False for j in range(target+1)] for i in range(k)]
    table[0][0] = True
    table[0][numbers[0]] = True
    # Construct the table
    for i in range(1,k):
        number = numbers[i]
        for j in range(target+1):
            table[i][j] = table[i-1][j]
            if j-number >= 0:
                table[i][j] = table[i-1][j] or table[i-1][j-number]
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Running time of this algorithm?

An algorithm for SUBSET SUM

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size(I) = instance size
= length (number of symbols) of reasonable encoding of instance *I*

Size of numbers

number(I)
= value of the largest number occuring in instance *I*

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Implication: the algorithm run in polynomial time when the numbers are encoded in *unary*-encoding, which requires the same number of symbols as the size of the number (e.g. tally-marks/"turven").

Questions?

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Definition

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if there exists an algorithm that solves instances I of X
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Which of the decision problems we studied so far is solvable in pseudo-polynomial time?

- SAT?
- 3-SAT?
- CLIQUE?
- IS?
- VC?
- Ex-Cov?
- SUBSET SUM
- PARTITION?
- HC?
- TSP?

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Observation: number(I) is only relevant for problems that involve numbers (distances, costs, weights, lengths, penalties, profits, time intervals, etc)

Which of the decision problems we studied so far is solvable in pseudo-polynomial time?

- SAT?
- 3-SAT?
- CLIQUE?
- IS?
- VC?
- Ex-Cov?
- SUBSET SUM
- PARTITION?
- HC?
- TSP?

Pseudo-polynomial time

Definition

A decision problem X is solvable in pseudo-polynomial time,
if there exists an algorithm that solves instances I of X
in time polynomially bounded in size(I) and number(I).

Observation: number(I) is only relevant for problems that involve numbers (distances, costs, weights, lengths, penalties, profits, time intervals, etc)

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Strong NP-hardness

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Theorem

If decision problem X
is strongly NP-hard and solvable in pseudo-polynomial time
then P=NP.

Are all number problems only weakly NP-hard?

THREE PARTITION

Instance: positive integers $a_1, \ldots, a_{3n}$ with $\sum_{i=1}^{3n} a_i = nA$
Question: does there exists a partition of the index set $\{1, \ldots, 3n\}$
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Theorem

THREE PARTITION is strongly NP-complete.

Proof: proof in Garey-Johnson shows that $SAT \leq_p 3DM \leq_p 4 - PARTITION \leq_p 3 - PARTITION$
Where the instance I constructed in the proof of $3DM \leq_p 4 - PARTITION$ has $number(I) \leq 2^{16}|A|^4$.

Questions?

NP versus coNP

Co-NP

Recall:

Definition

A decision problem X lies in the complexity class NP,
if the YES-instances of X possess certificates of polynomial length
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A decision problem X is NP-complete,
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Problems in coNP

Non-HAMILTONICITY

Instance: an undirected graph $G = (V, E)$

Question: is G not Hamiltonian?

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Instance:

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Instance: a set of logical variables $X := \{x_1, \ldots, x_n\}$ and a logical formula Φ in disjunctive normal form (DNF) over X

Question: are all truth settings for X satisfying for Φ?

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Theorem

Non-HAMILTONICITY, UNSAT and TAUTOLOGY are coNP-complete.

Problems in coNP

Theorem

Non-HAMILTONICITY, UNSAT and TAUTOLOGY are coNP-complete.

Lemma

If X is NP-complete, $\bar{X}$ is coNP-complete.

$\Rightarrow$ NP-completeness of Non-HAMILTONICITY & UNSAT

Questions?

Excursion: Logical formulas

Let X be a set of logical *variables*.

- *Truth assignment*: $t : X \to \{true, false\}$
- *Literals:* We call x and $\neg x$ literals corresponding to variable $x \in X$. $t(\neg x) = true \Leftrightarrow t(x) = false$
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- logical formula in conjunctive normal form (CNF):
  conjunction of disjunctive clauses, e.g. $(x1 \vee \neg x_2) \wedge (x_1 \vee x_3)$
- logical formula in disjunctive normal form (DNF):
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Satisfiability (SAT) - as we use it / CNF-SAT

Instance: set of logical variables $X := \{x_1, \ldots, x_n\}$, logical formula Φ in CNF
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DNF-SAT

Instance: set of logical variables $X := \{x_1, \ldots, x_n\}$, logical formula Φ in DNF
Question: does there exist a truth assignment for X that satisfies Φ?
In P (as the formula can be interpreted as an enumeration of solutions)

De Morgan's law

- $\neg(l_1 \wedge l_1) = \neg l_1 \vee \neg l_2$
- $\neg(x \vee y) = \neg x \wedge \neg y$

This implies that the negation of a CNF formula is a DNF formula and vice versa.

TAUTOLOGY is coNP-complete

TAUTOLOGY

Instance: a set of logical variables $X := \{x_1, \ldots, x_n\}$ and DNF-formula Φ over X
Question: are all truth settings for X satisfying assignments for Φ?

Theorem

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Proof:

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Questions?

NP versus coNP

Problems in $NP \cap coNP$ have

- good certificates for YES-instances
- good certificates for NO-instances

Example

MaxFlow (LP):

Instance: a directed graph $G = (V, E)$ with a dedicated source node s and sink node t and edge capacities c_e for all $e \in E$, a flow value F

Question: Is there an s-t-*flow* of size at least F?

That is: is there an assignment $f : E \to \mathcal{R}$ such that
$\sum_{e \text{ outgoing edge of } s} f_e = F$,
$\sum_{e \text{ ingoing edge of } t} f_e = F$,
$\sum_{e \text{ ingoing edge of } t} f_e - \sum_{e \text{ outgoing edge of } v} f_e = 0$?

The Soviet railway system problem

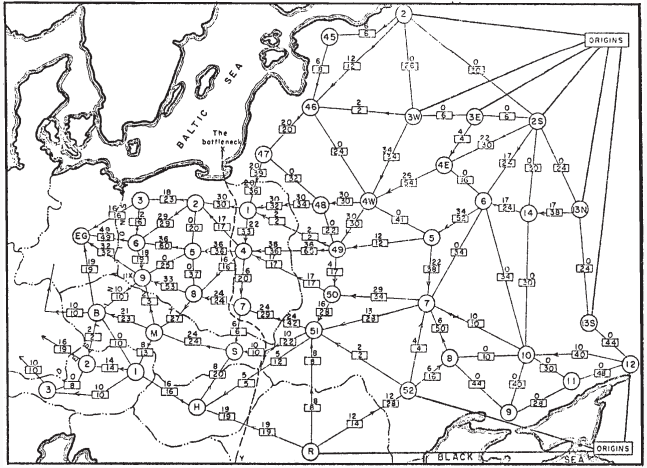

Fig. 2. From Harris and Ross [11]: Schematic diagram of the railway network of the Western Soviet Union and Eastern European countries, with a maximum flow of value 163,000 tons from Russia to Eastern Europe, and a cut of capacity 163,000 tons indicated as "The bottleneck"

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MaxFlow in NP

NP versus coNP (3)

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MaxFlow in coNP (min-cut/max-flow theorem)

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MaxFlow in NP
MaxFlow in coNP (min-cut/max-flow theorem)
MaxFlow is in P (Ford-Fulkerson algorithm)

NP versus coNP (3)

Example

Linear Programming (LP): Instance: a matrix A; vectors c and b; a bound t

Question: does there exist a real vector x with $Ax \leq b$ and $cx \leq t$?

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LP-duality tells us that if and only if there is a y with $A^t y = c$ and $b^t y > t$, then the answer is 'NO'.

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- LP lies in coNP

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LP lies in P (see, e.g., Karmakar's algorithm).

Is $P = NP \cap coNP$?

- FACT: $P \subseteq NP \cap coNP$

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Problems that are in $NP \cap coNP$ but not known to be in P are rare.

Two famous examples are *Graph Isomorphism* and *Integer Factorization*.

Questions?

Example

Integer Factorization (version a):
Instance: integers y, u (given in binary).
Question: Is there an integer x that divides y and satisfies $1 < x \leq u$?

Version a) is in NP

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Note: basic arithmetic (division, multiplication) is in polynomial time. Primality testing is in P.

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Many cryptographic protocols are based on the difficulty of factoring large composite integers - an algorithm that efficiently factors an arbitrary integer would render these insecure.

Example

Integer Factorization (version b):
Instance: integers y, l, u (given in binary).
Question: Is there an integer x that divides y and satisfies $l \leq x \leq u$?

Version b) is in NP.
Is it also in coNP?

It is not straightforward to give a NO-certificate.

Example

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The integer factorization $\Pi_{i=1,\ldots m} p_i$ with p_i primes (not necessarily different ones) does not (directly) give us a certificate.

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Idea: work with the integer factorization, and then consider for each cardinality $1, \ldots, m$ the possible sets. For a fixed cardinality the sets are nicely ordered.

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Any ideas?
Idea: work with the integer factorization, and then consider for each cardinality $1, \ldots, m$ the possible sets. For a fixed cardinality the sets are nicely ordered. Then construct a certificate for each cardinality separately.

NP versus coNP

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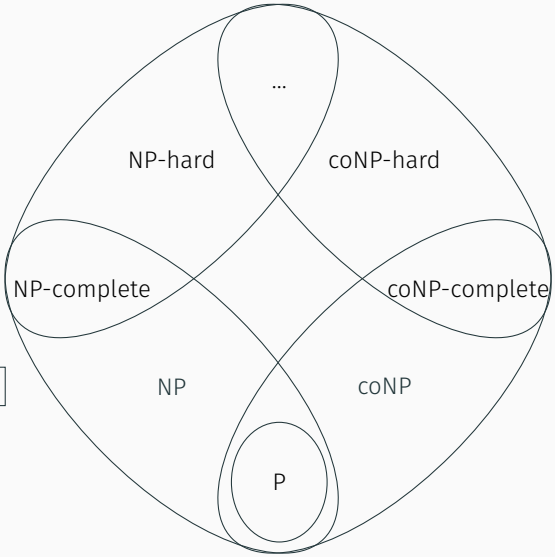
- FACT: If $P = coNP$ then $P = NP$ (P closed under complementation)

Complexity Classes



If $P \neq NP$, $NP \neq coNP$

Questions?

Beyond NP and coNP

A short note on other complexity classes

The class NP relates to questions with polynomial proofs of $\exists x \in I : \text{YES}(x)$ for a solution x of an instance I.

The class coNP relates to questions with polynomial proofs of $\forall x \in I : \neg\text{YES}(x)$

Polynomial Hierarchy and Complexity classes $\Sigma_k P$ and $\Pi_k P$

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$\exists x_1, \forall x_2, \exists x_3, \ldots (\forall \text{ or } \exists) x_k : A(x_1, \ldots, x_k)$

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The polynomial hierarchy class PH is the union of all these set for all k.

NP = $\Sigma_1 P$ and coNP = $\Pi_1 P$.

If coNP = NP, this would imply that $\Sigma_k P$ = NP and $\Pi_k P$ = coNP for all $k \geq 1$ (this is not expected).

The class $\Sigma_2 P$ relates to robust programming, stochastic programming, Stackelberg games, etcetera.

Gerhard Woeginger gave an excellent keynote on this topic at EURO2018 (video link).

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It is known that $P \subset EXPTIME$ and $PSPACE \subset EXPSPACE$

An unsolvable decision problem

Problem: CheckTermination (also called ‘Halting Problem’)

Input: two text pieces *`text1`* and *`text2`*

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What does CT(text4,text4) do if *text4* is the Python code of *wrong*???

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What does CT(text4,text4) do if *text4* is the Python code of *wrong*???
Conclusion: There is no algorithm for CheckTermination

Technique is called *diagonalization.*

Questions?