Algorithms and Complexity (AC)

Fourth Lecture (2nd of October, 2023)

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LNMB – Landelijk Netwerk Mathematische Besliskunde

Table of contents

1. Basic definitions

2. Ad-hoc approaches

   Vertex cover

   Makespan minimization

   Intermezzo: Euler tours

   Travelling Salesman

3. Linear Programming-based approaches

   Weighted vertex cover

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Let X be a minimization problem.

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We aim for polynomial time algorithms with good approximation ratios
This is still possible for many problems whose decision versions are NP-complete!

Questions?

Ad-hoc approaches

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How can we find a 'good' vertex cover?

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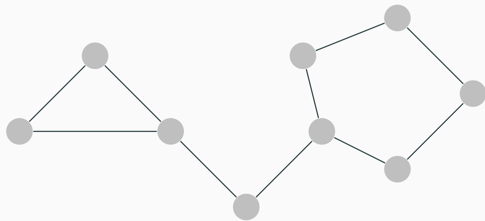

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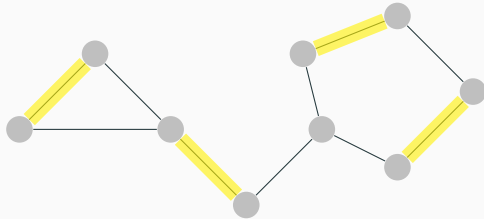

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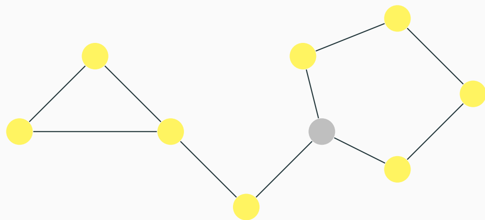

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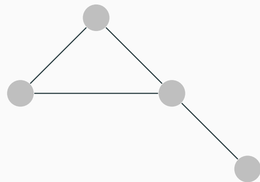

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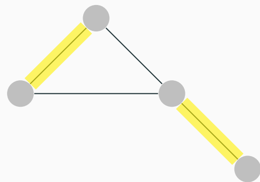

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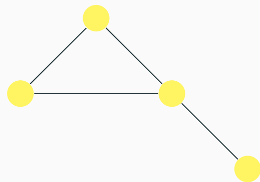

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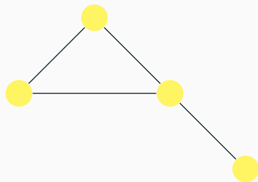

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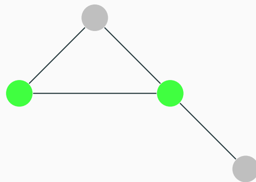

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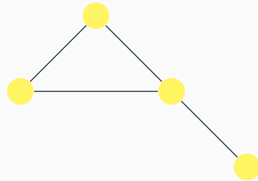

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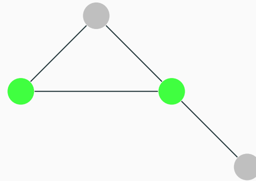

Note: This is the best know approximation algorithm for vertex cover. But it is an open problem whether a better one exists.

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Instance: m machines; n jobs with processing times $p_1, \dots, p_n$
Goal: assign jobs to machines so that the maximum workload (= makespan) is minimized

Decision variant in NP, NP-hard (thus NP-complete) already for $m = 2$:

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Can sharpen above analysis to guarantee $2 - 1/m$ (exercise).

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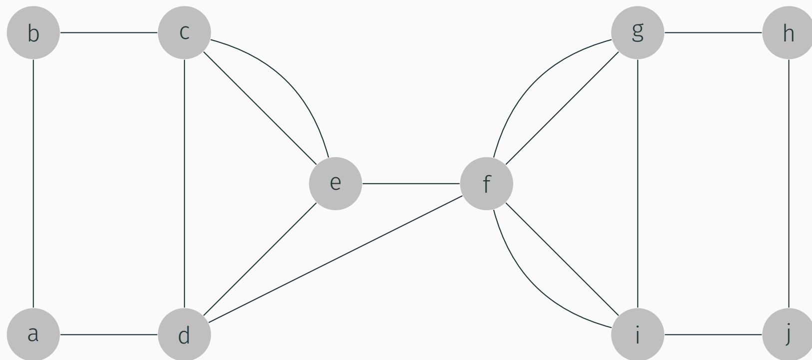
- Multi-graph: $G = (V, E)$ but now E may be a multi-set (edges may occur multiple times)
- A Eulerian tour of a graph is a tour visiting all *edges* exactly once.



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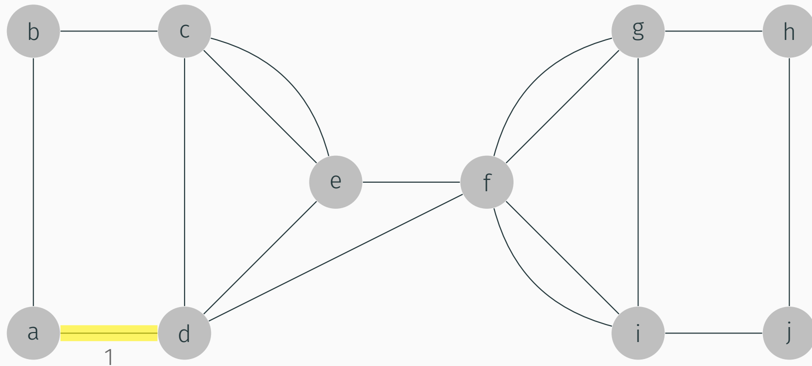

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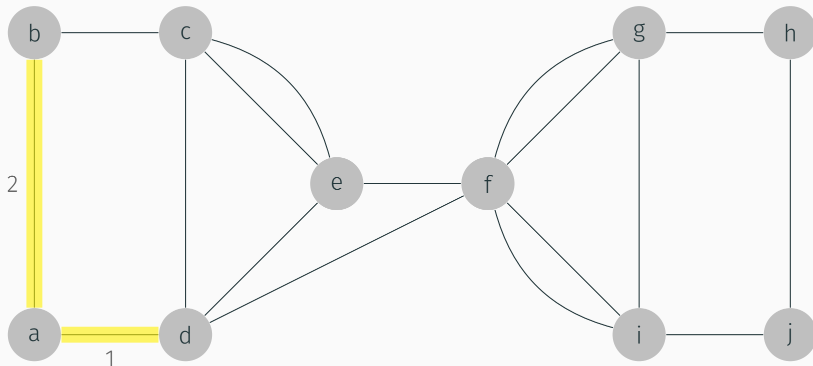

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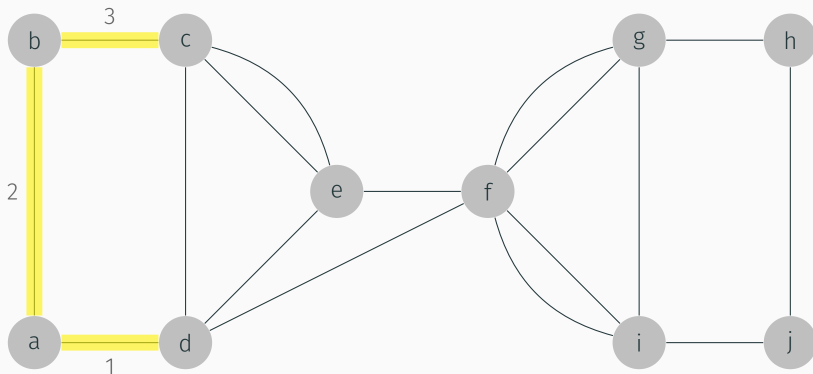

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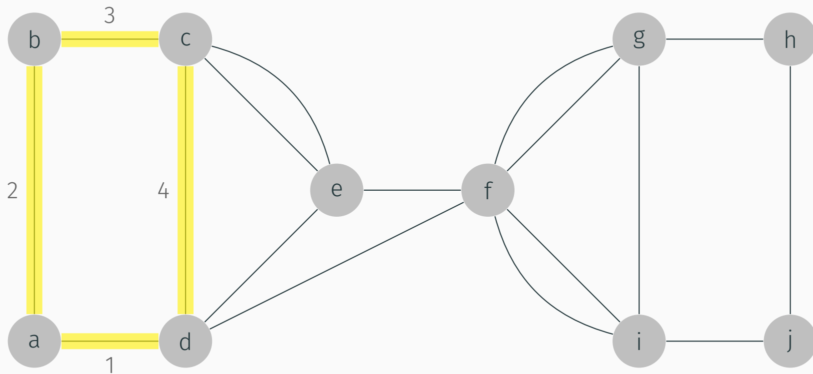

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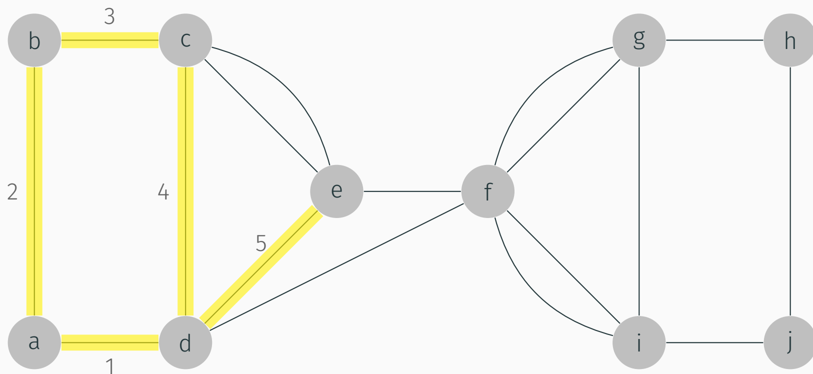

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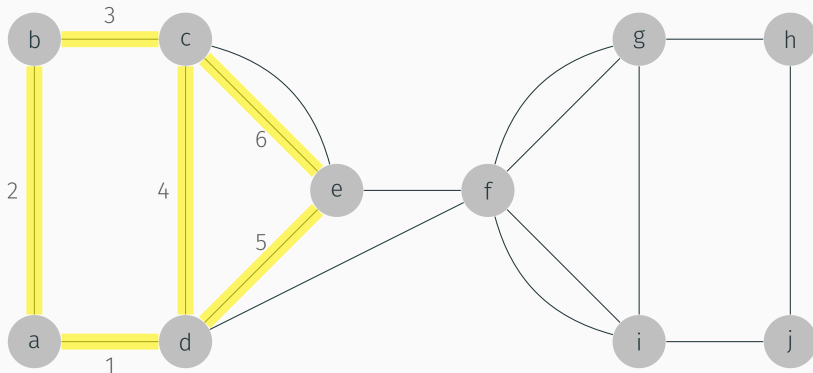

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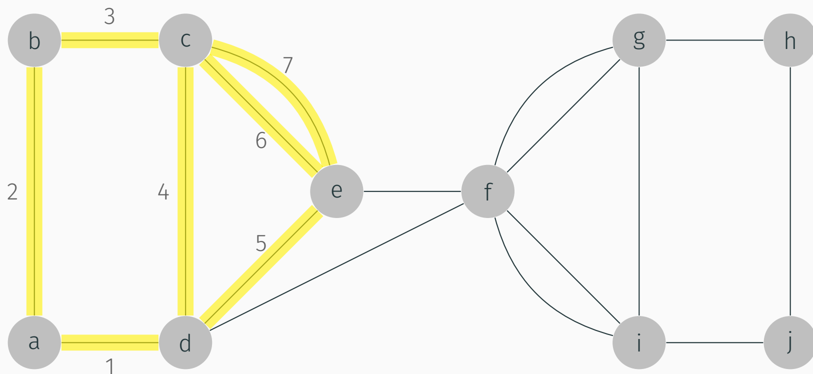

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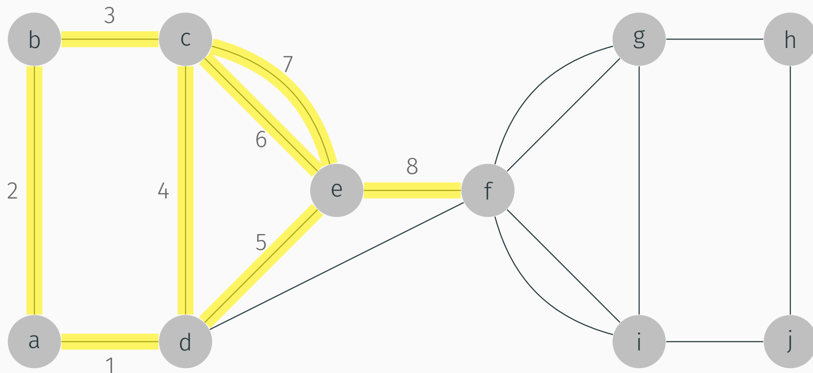

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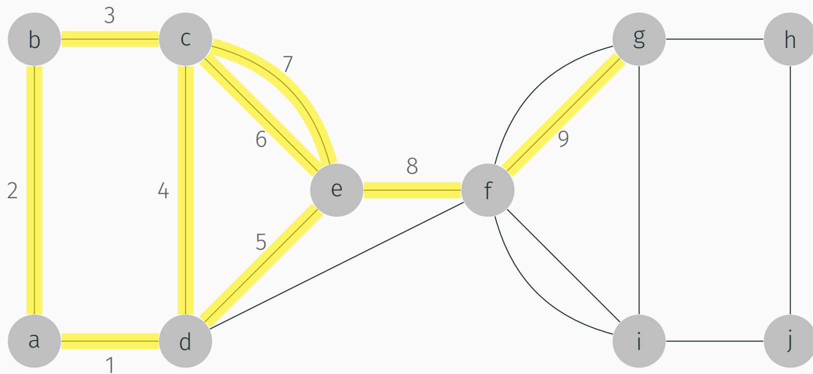

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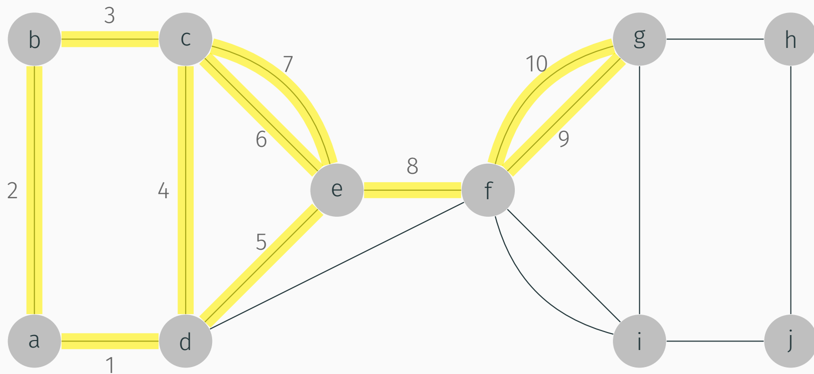

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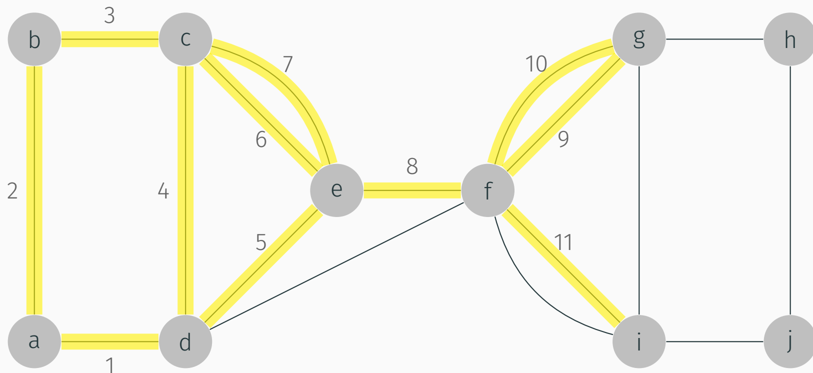

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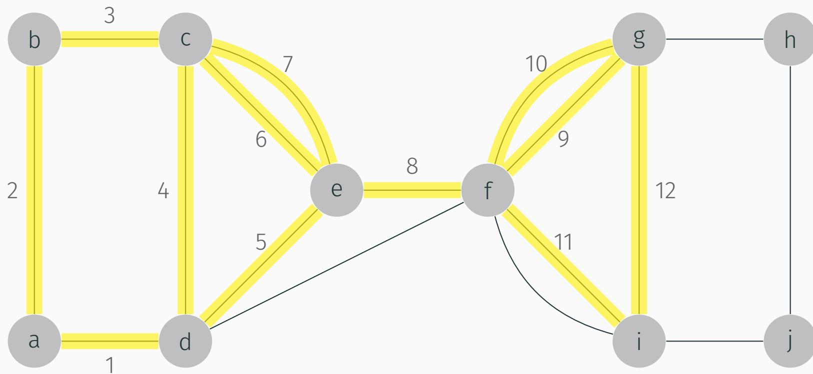

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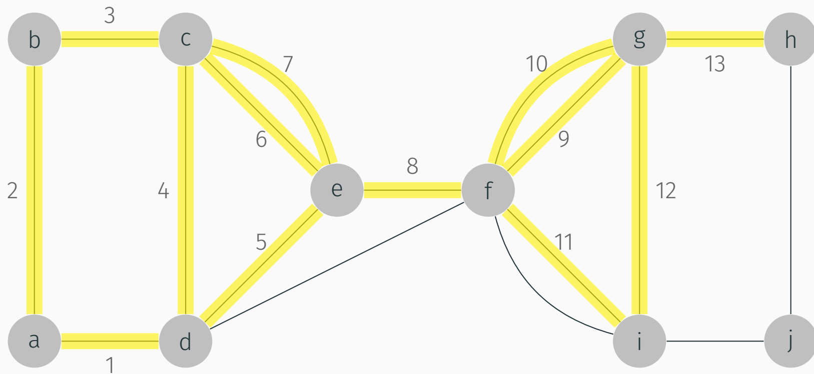

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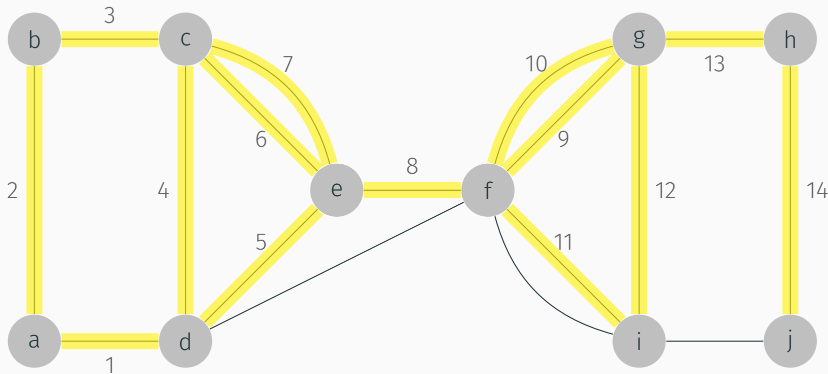

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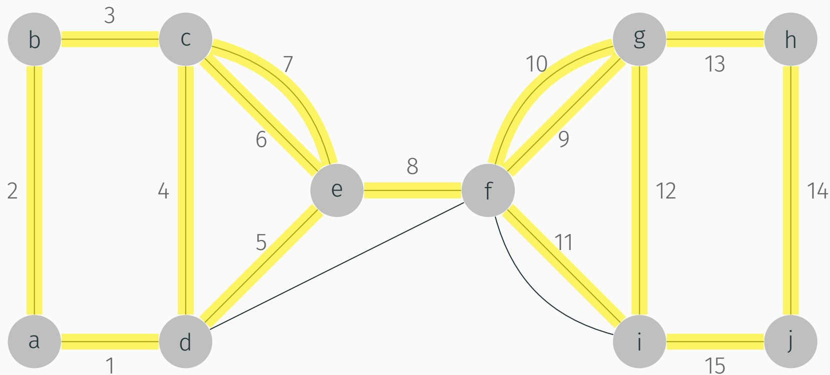

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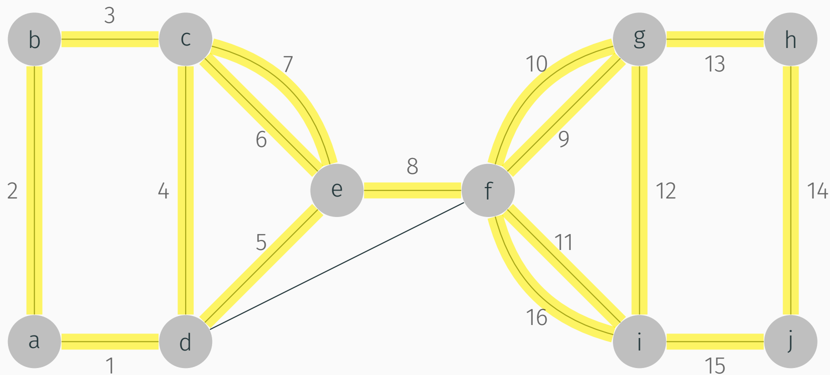

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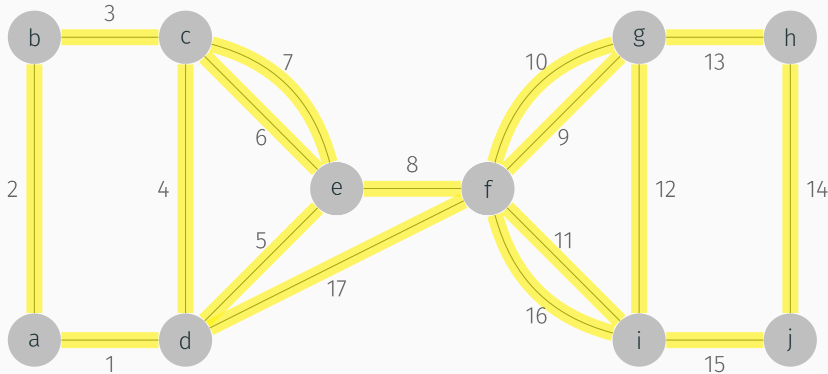

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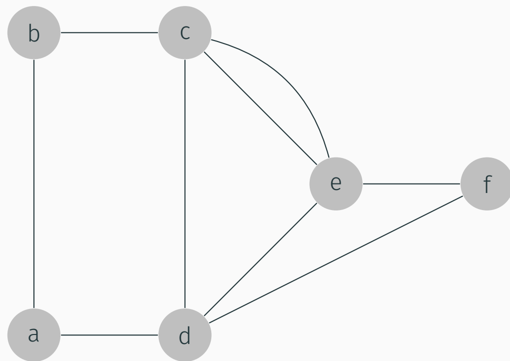
(This is a constructive polynomial-time algorithm)

Intermezzo: Euler tours



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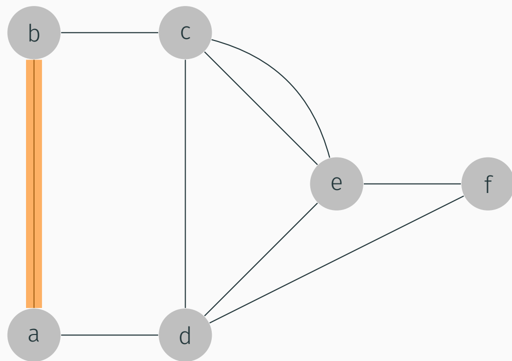

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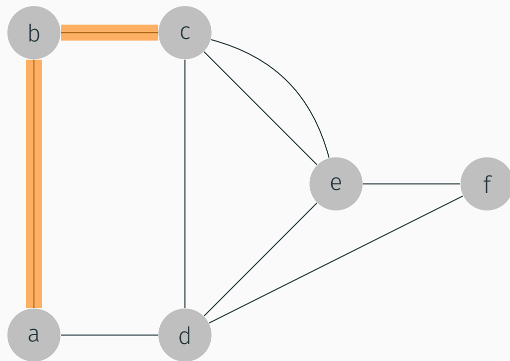

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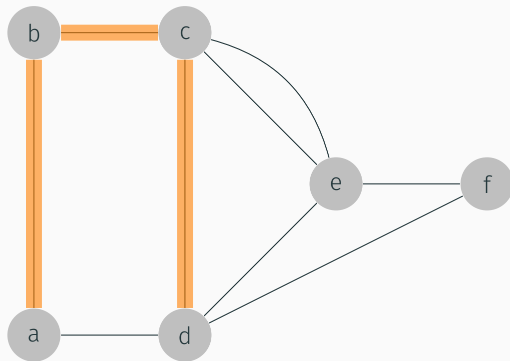

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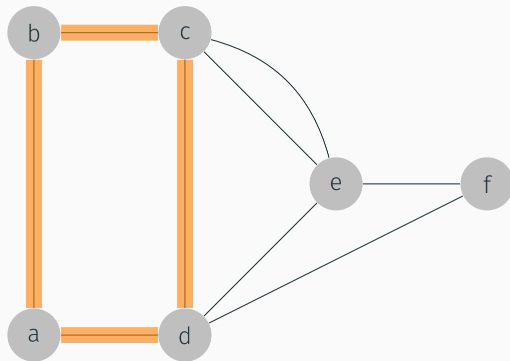

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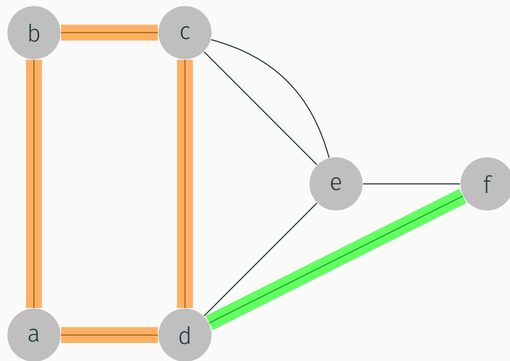

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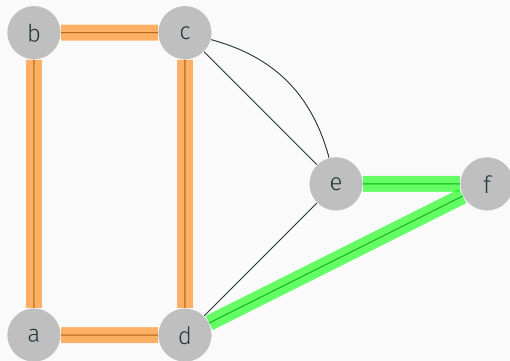

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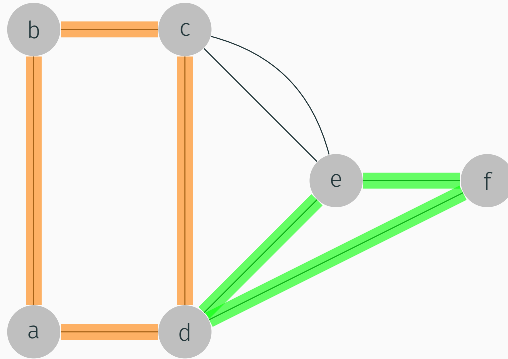

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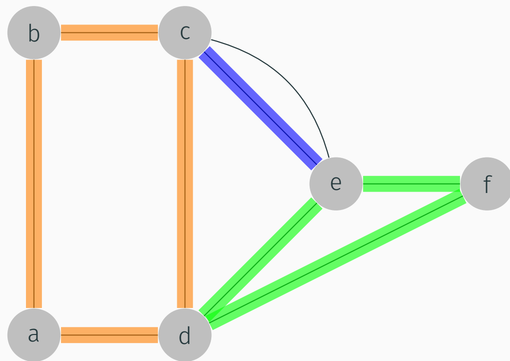

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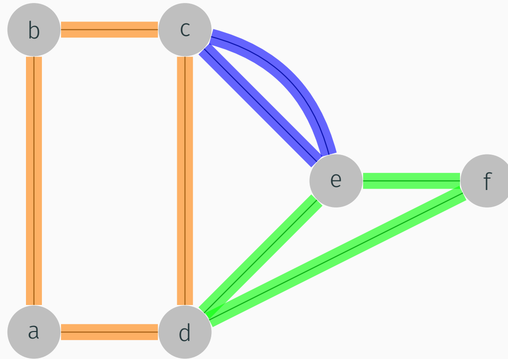
Intermezzo: Euler tours



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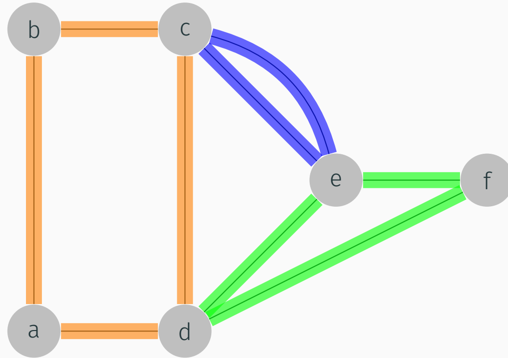

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Observe: every sub-walk can only end when it returns to the starting vertex (due to the even-degree property)

Questions?

TSP (Optimization version)

Instance: cities $1, \dots, n$; distances $d(i, j)$

Goal: find roundtrip of smallest possible length

NP-complete (reduction from Hamiltonian cycle)

Can we approximate the TSP?

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- $\text{opt}(I) \geq$ twice the length of min. weight perfect matching, if n even

Double-tree algorithm

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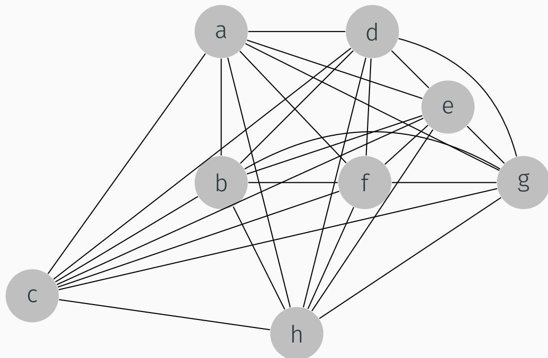

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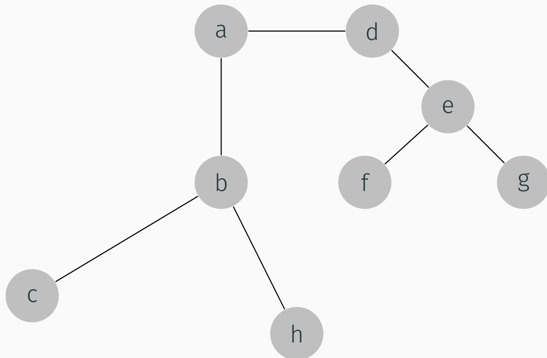

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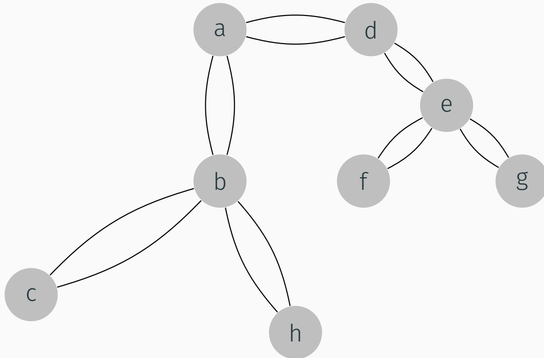

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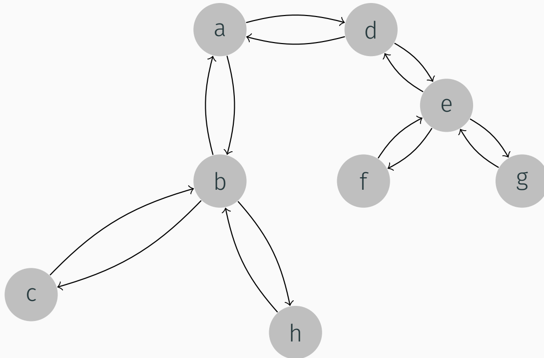

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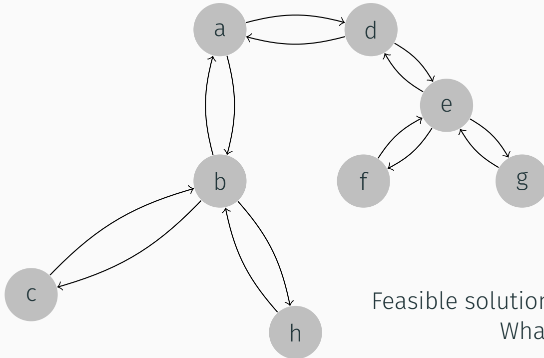
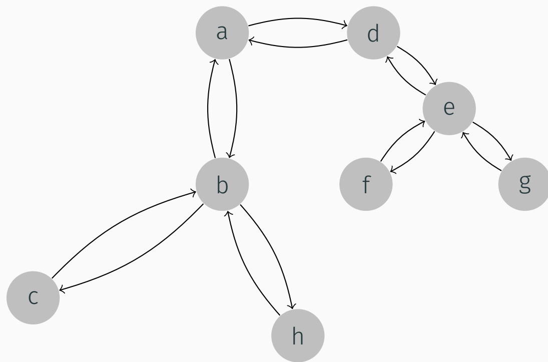
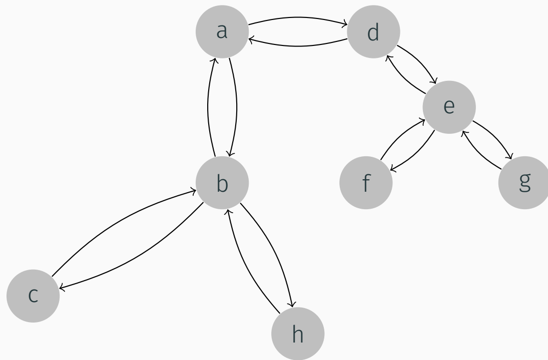


Feasible solution if cities may be visited more than once.
What is the approximation ratio?



Approximation ratio of double tree algorithm



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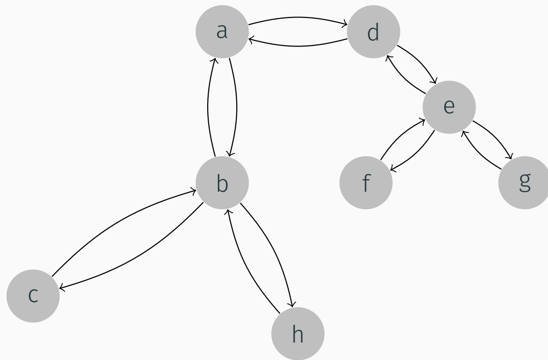
$\text{opt}(I) \geq \text{MST}(I)$



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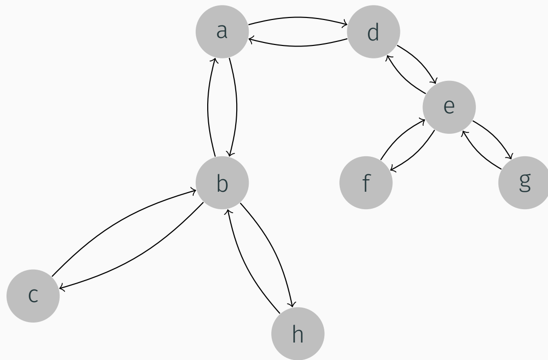

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$\Rightarrow \sup_I \frac{A(I)}{\text{opt}(I)} \leq \frac{2\text{MST}(I)}{\text{MST}(I)} = 2$

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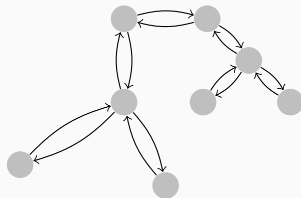

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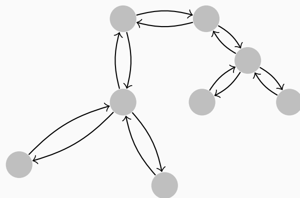

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If the distances satisfy the triangle inequality $d(x, y) + d(y, z) \;\geq\; d(x, z)$ for all cities x, y, z

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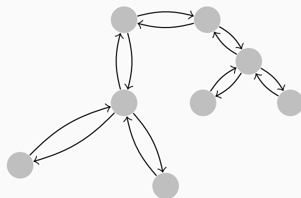

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All are NP-complete. For metric TSP reduce from HC using distances 1 and 2)

Euclidean-TSP (points in plane) requires harder reduction (beyond scope for us)

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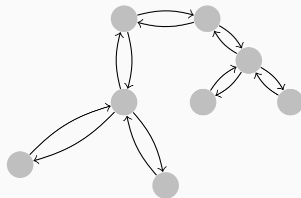

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In case of Metric TSP, we can “shortcut”!

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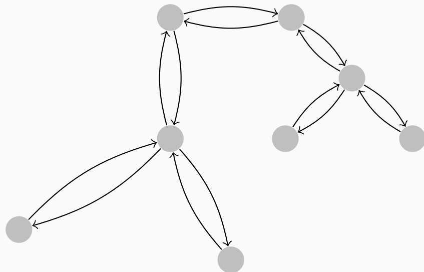

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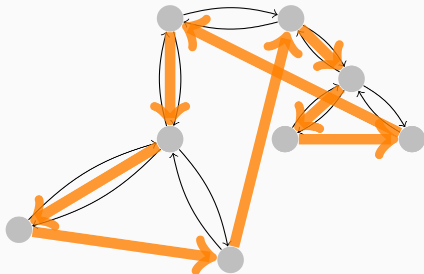

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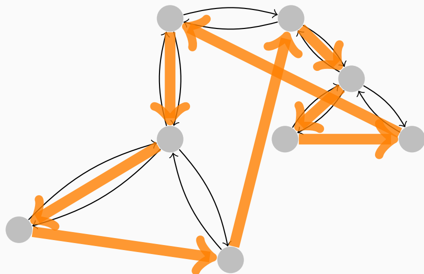

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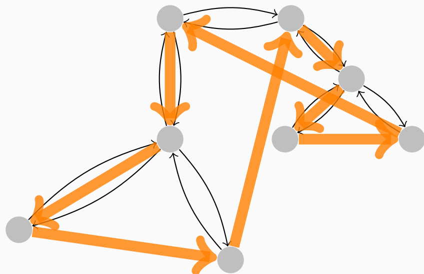

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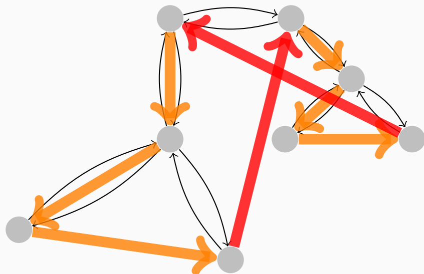

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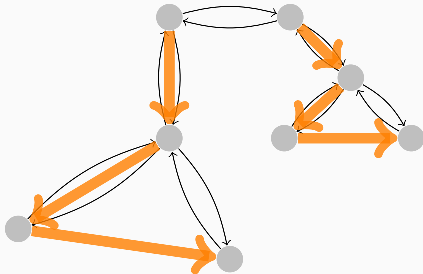

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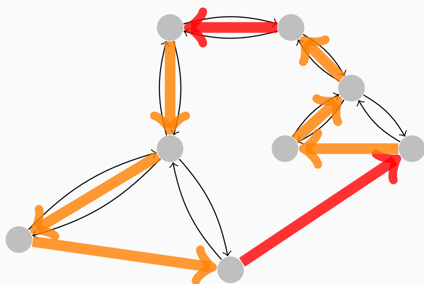

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Christofides Algorithm

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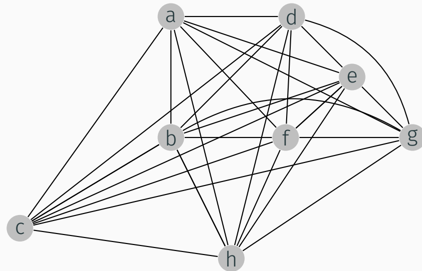

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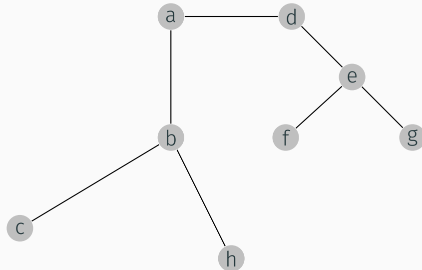

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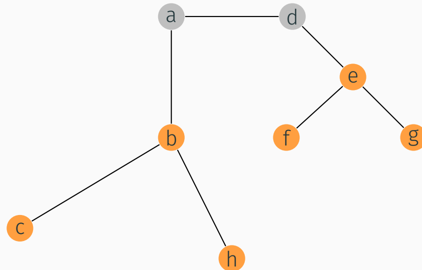

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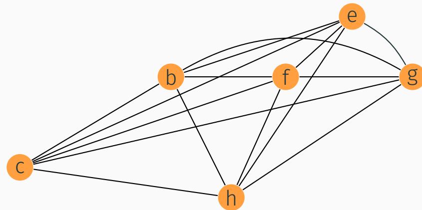

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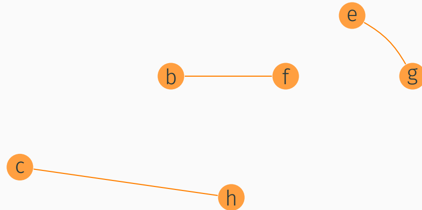

Christofides Algorithm

Algorithm

- Represent cities and distances as a graph with edge weights
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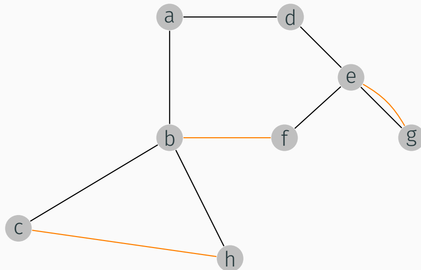

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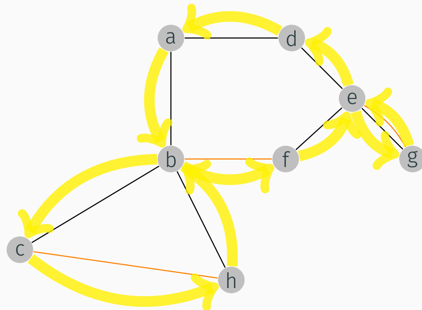

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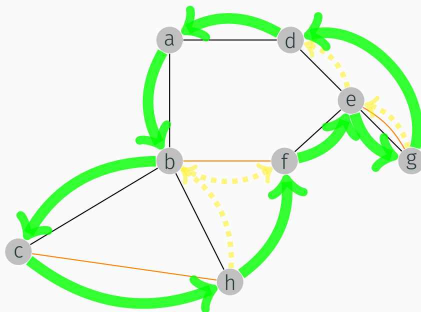
- Shortcut the Euler tour to a TSP tour



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Odd-degree vertices

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Original proof due to seminal work by Jack Edmonds (1965), Maximum matching and a polyhedron with 0,1-vertices, *Journal of Research National Bureau of Standards Section B 69*.

Many later references/survey articles available. E.g. *Computing minimum-weight perfect matchings* by Cook and Rohe, INFORMS Journal on Computing 1999.

Christofides Algorithm

Theorem

The Christofides algorithm has approximation ratio 1.5.

Proof:

Consider the following lower bounds:

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Final Remark

Since recently the algorithm is sometimes referred to as Christofides–Serdyukov.

See *A historical note on the 3/2-approximation algorithm for the metric traveling salesman problem* (2020) in Historica Mathematica

Questions?

Linear Programming-based approaches

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LP-based approaches

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3. Solve the LP relaxation in polynomial time
4. Round the optimal LP solution to approximate ILP solution (preserving feasibility!)

Weighted vertex cover (VC)

Instance: a graph $G = (V, E)$; weights $w : V \to \mathbb{R}^+$
Goal: find a vertex cover of smallest possible weight
(e.g. find a vertex cover $X \subseteq V$ minimizing $\sum_{v \in V} w(v)$)

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ILP formulation

$$\begin{aligned} \text{minimize} \quad & \textstyle\sum_{v \in V} w(v) \cdot x_v \\ \text{subject to} \quad & x_u + x_v \geq 1 \quad \text{for every edge } \{u, v\} \in E \\ & x_v \in \{0, 1\} \quad \text{for every vertex } v \in V \end{aligned}$$

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This polynomial-time approximation algorithm has approximation ratio 2.

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