Algorithms and Complexity: Turing machines

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(Based on slides by Gerhard Woeginger and Jesper Nederlof)

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LNMB

Complexity classes P and NP

Definition: Complexity class P

A decision problem X lies in the complexity class P,

- **if it can be solved on a deterministic Turing machine in polynomial time** (original, formal definition)
- (or, alternatively:) if it is solved by an algorithm with polynomial time complexity (definition that we use)

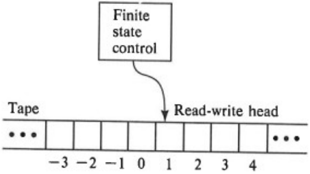
Definition: Complexity class NP

A decision problem X lies in the complexity class NP, if

- **if it can be solved in polynomial time on a non-deterministic Turing machine** (original, formal definition)
- (or, alternatively:) if it is solved by a *non-deterministic* algorithm with polynomial time complexity.
- (or, alternatively:) if the YES-instances of X possess certificates of polynomial length that can be verified in polynomial time.

Turing machines

Not talking about this.



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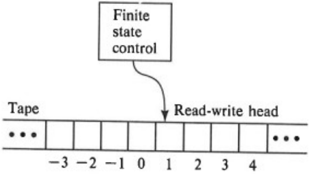
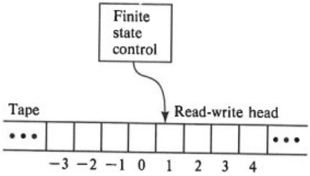


But this!
$\rightarrow$ Alternative model of computation

Deterministic one-tape Turing machine (DTM)

A DTM consists of

1. a *finite state control*
2. a *read-write head*
3. a *tape*: two-way infinite sequence of tape squares



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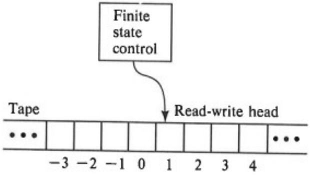

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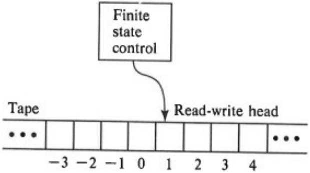

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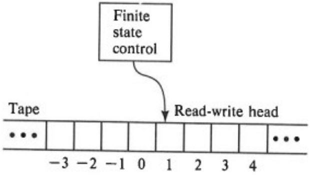
2. a finite set Q of *states*, inclusing a distinguished *start state* q_0 and two distinguished *halt states* q_Y and q_N
3. a *transition function* $\delta : (Q \setminus \{q_Y, q_N\}) \times \Gamma \to Q \times \Gamma \times \{-1, 1\}$



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end while
if $q = q_Y$ **then**
  return YES
else
  return NO
end if

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Each iteration of the while-loop counts as a step

Example: A program for a DTM machine

$\Gamma = \{0, 1, b\}$, $\Sigma = \{0, 1\}$, $Q = \{q_0, q_1, q_2, q_Y, q_N\}$

q	0	1	b
q_0	$(q_0, 0, +1)$	$(q_0, 1, +1)$	$(q_1, b, -1)$
q_1	$(q_2, b, -1)$	$(q_3, b, -1)$	$(q_N, b, -1)$
q_2	$(q_Y, b, -1)$	$(q_N, b, -1)$	$(q_N, b, -1)$
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- What does this program do?
- Try it out on different strings, e.g., 0111, 010, 11110, 000000, ...
- When does it return 'yes' and when 'no'?
- How many steps do we need in these examples?
- How many steps do we need at most for a string of length n?
- How much space do we need at most for a string of length n?

Would you rather own a RAM or a DTM?

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Equivalence of computational models

A RAM and a DTM are equivalent in the sense that any function that can be computed on a DTM can be computed on a RAM, and vice versa.

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Church-Turing thesis

Anything that can be calculated by an *effective method* can be computed by a deterministic Turing machine.

Complexity classes P and NP

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A decision problem X lies in the complexity class P,

- **if it can be solved on a deterministic Turing machine in polynomial time** (original, formal definition)
- (or, alternatively:) if it is solved by an algorithm with polynomial time complexity (definition that we use)

Non-deterministic Turing machine

Non-deterministic Turing machine (NDTM)

1. guessing module: write-only head
2. checking module: deterministic Turing machine

Non-deterministic Turing machine

A program for a NDTM specifies:

exactly the same as a DTM program:

1. finite set of tape symbols Γ of tape symbols, including blank symbol
2. finite set Q of *states*, i
3. *transition function* δ

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Non-deterministic Turing machine

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- write input string in tape squares 1 to $|x|$
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Note: For a given string x and a given NDTM program, there is an *infinite* number of possible computations possible (one for each 'guessed' string)

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Terminology & definitions

Accepting computation: all computations that terminate in accepting state (q_Y).
Non-accepting computations: all computations that terminate in non-accepting-state (q_N) or do not terminate at all.

NDTM program M **accepts** x if *there is* an accepting computation for x on M.

The **time complexity** of an NDTM program for a string x is defined as the *minimum* running time over all accepting computations of x by M.
The worst-case time-complexity of an NDTM program is the maximum time complexity over all strings x of a certain length n that are accepted by M.

Non-deterministic algorithm

non-deterministic algorithm $\hat{=}$ program for a non-deterministic Turing machine

1. Oracle/guessing stage
2. Checking stage

time complexity of a non-deterministic algorithm
$\hat{=}$ time complexity of the corresponding program

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Definition: Complexity class NP

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Warnings:

1. A non-deterministic Turing machine is a *theoretical* construct, not an actual machine!
2. The Church-Turing thesis relates to *deterministic* Turing machines. (*'Anything that can be calculated by an effective method can be computed by a 'deterministic' Turing machine.'*