Algorithms and Complexity

Lecture 7

Dynamic Programming and
Inclusion Exclusion

Overview of Today

- Dynamic Programming
  - Subset Sum / Knapsack approximation
  - Coloring
  - Traveling Salesman Problem
  - Steiner Tree
  - Weighted Independent Set on trees
- Inclusion Exclusion
  - Hamiltonian Cycle
  - k-coloring

Dynamic Programming (DP)

- The method works with a big table of data that has to be stored in memory.
- A relatively easy procedure computes new table entries using already computed table entries.
- This easy procedure is often so easy that we just write it down as a single formula, obtaining a *recurrence*.

- Other interpretation:
  - cleverly define subproblems,
  - solve easier subproblems first; store solutions
  - use solutions of easier subproblem to solve harder ones
  - the original problem also is a subproblem

Subset Sum

- Given integers $w_1, \ldots, w_n, t$ find $X \subseteq \{1, \ldots, n\}$ such that $\sum_{i \in X} w_i = t$.
- For $i = 1, \ldots, n$ and $j = 1, \ldots, t$ define $A[i, j]$ to be true iff $\exists X \subseteq \{1, \ldots, i\}$ s.t. $\sum_{i \in X} w_i = j$. Want: $A[n, t]$
- Instance {1 2 3 4 5 6 7 8 9 10 11 12}, t= 50
- True/false: A[0,0],A[2,0],A[3,5],A[6,12],A[4,20],A[12,50]?

$$A[i,j] = \begin{cases} \textbf{false} & \text{if } i = 0, j > 0, \\ \textbf{true} & \text{if } i = 0, j = 0, \\ A[i-1,j] & \text{if } i > 0,\ j < w_i, \\ A[i-1,j] \vee A[i-1,j-w_i] & \text{otherwise.} \end{cases}$$

	0	1	...	n
0				
1				
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9				
10				
...				
t				?

Algorithm $\mathtt{sss}(w_1, \ldots, w_n, t)$
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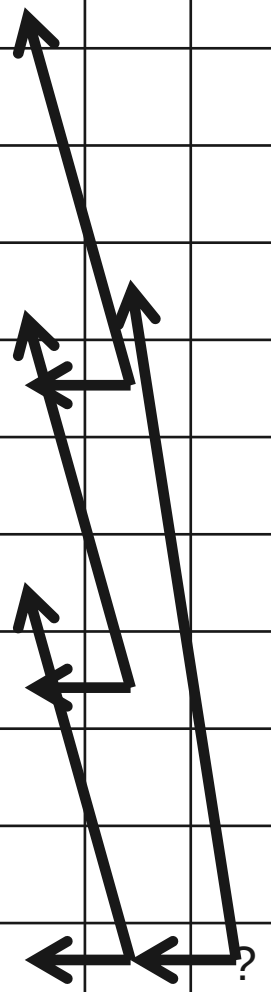

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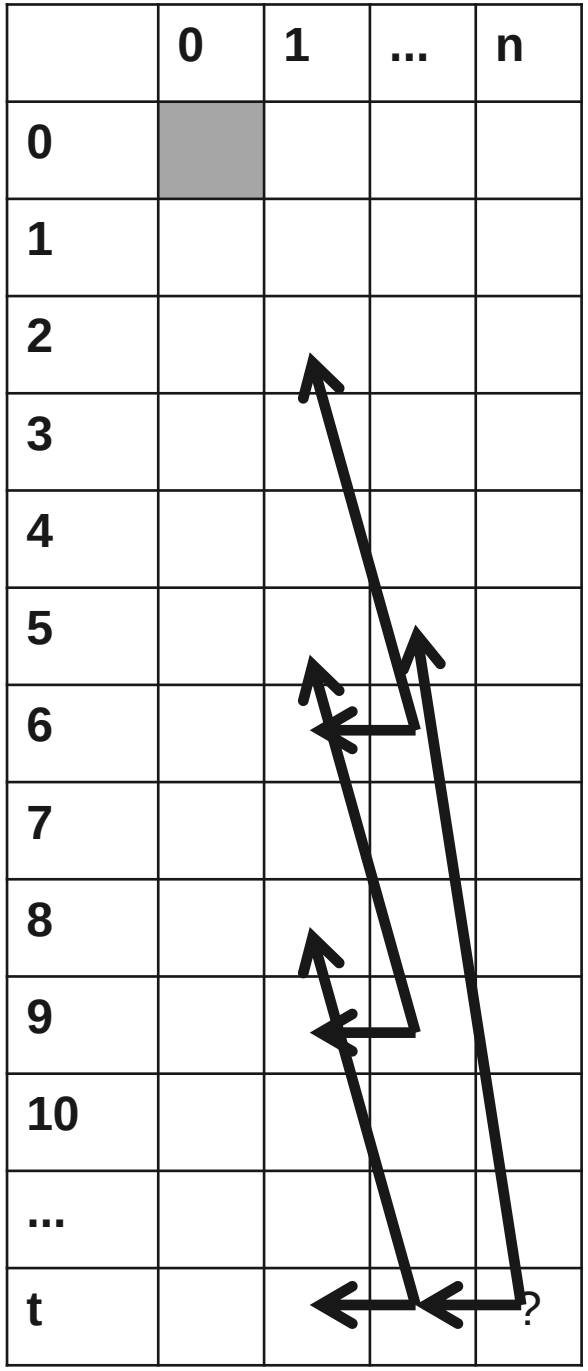

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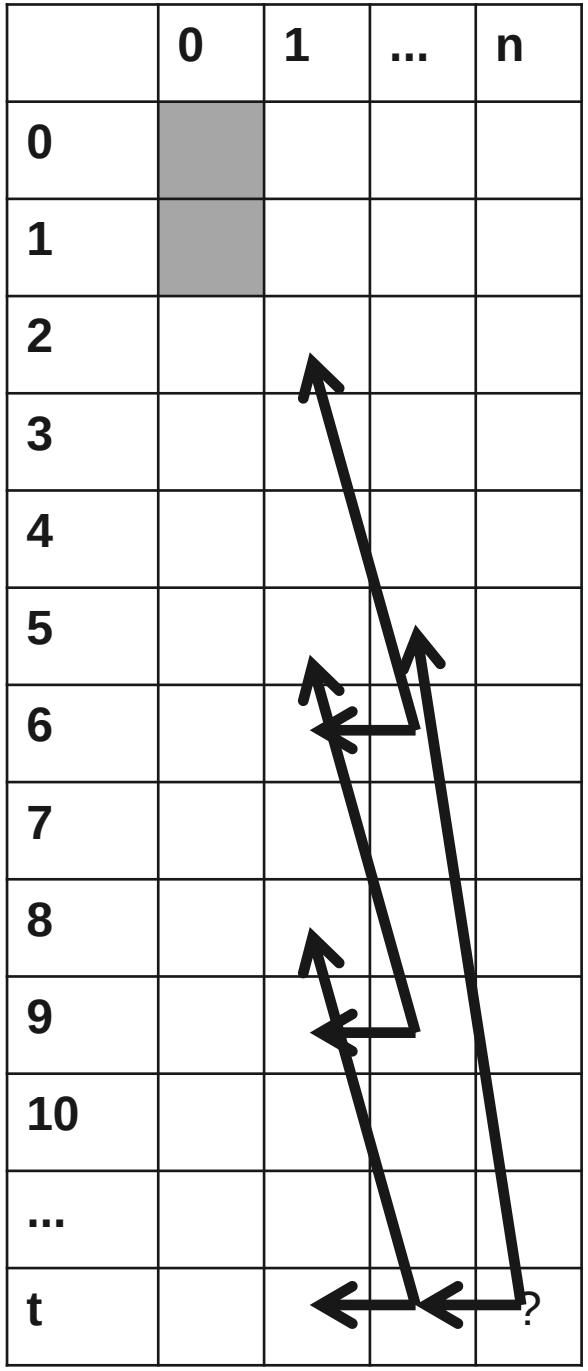

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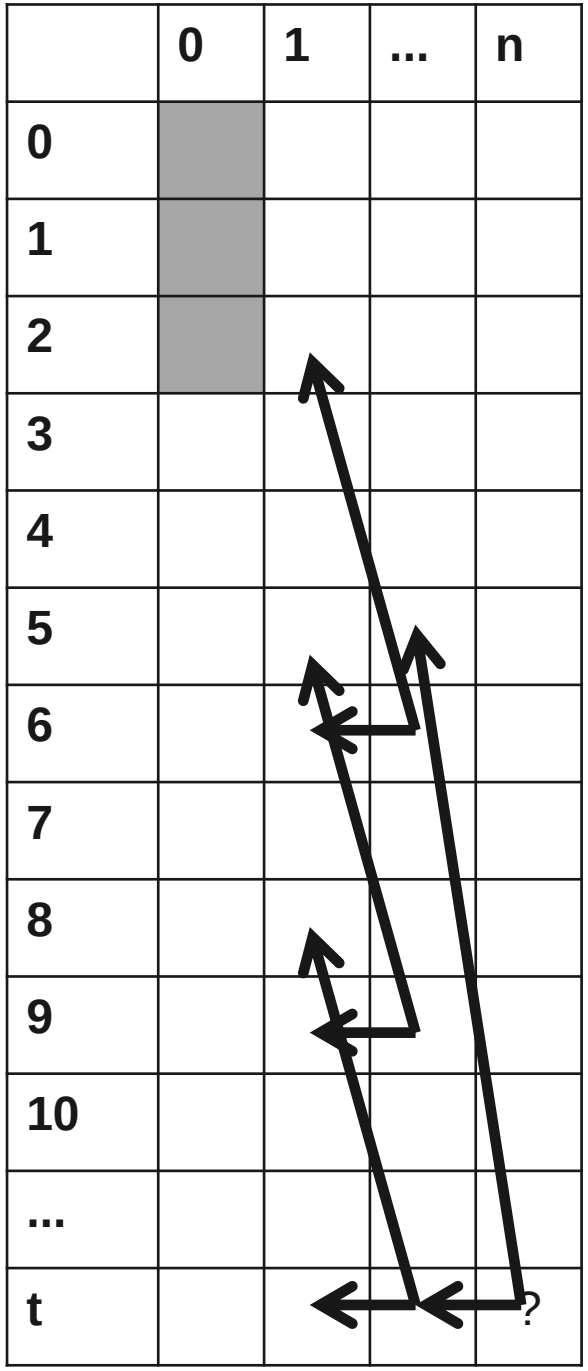

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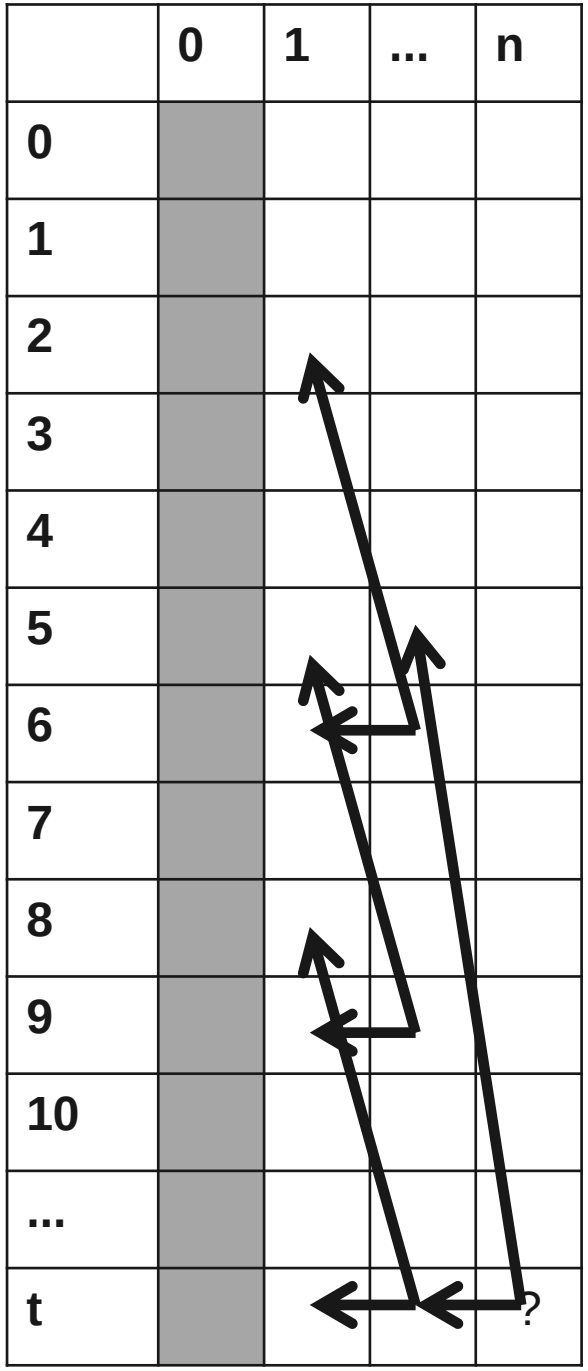

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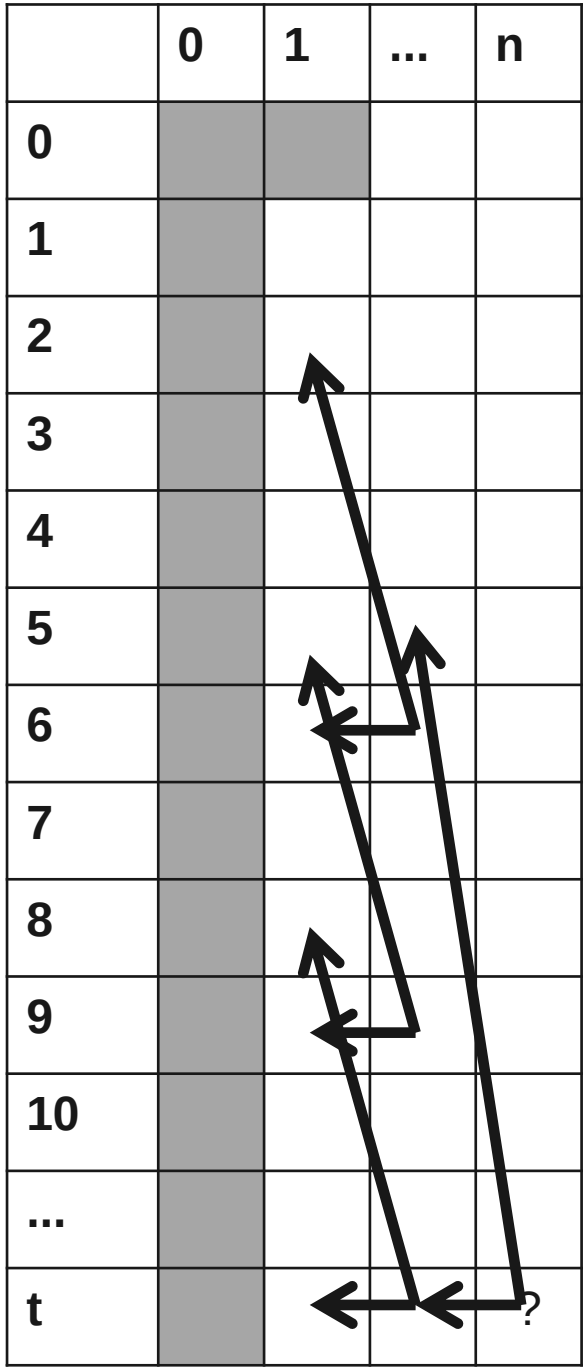

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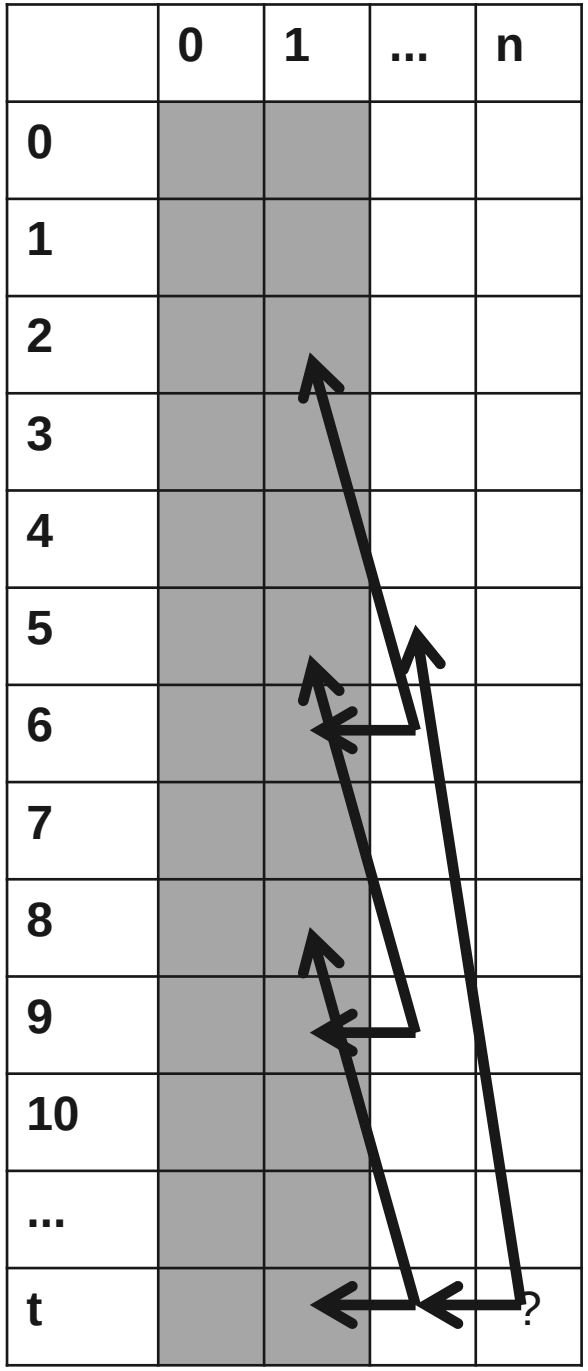

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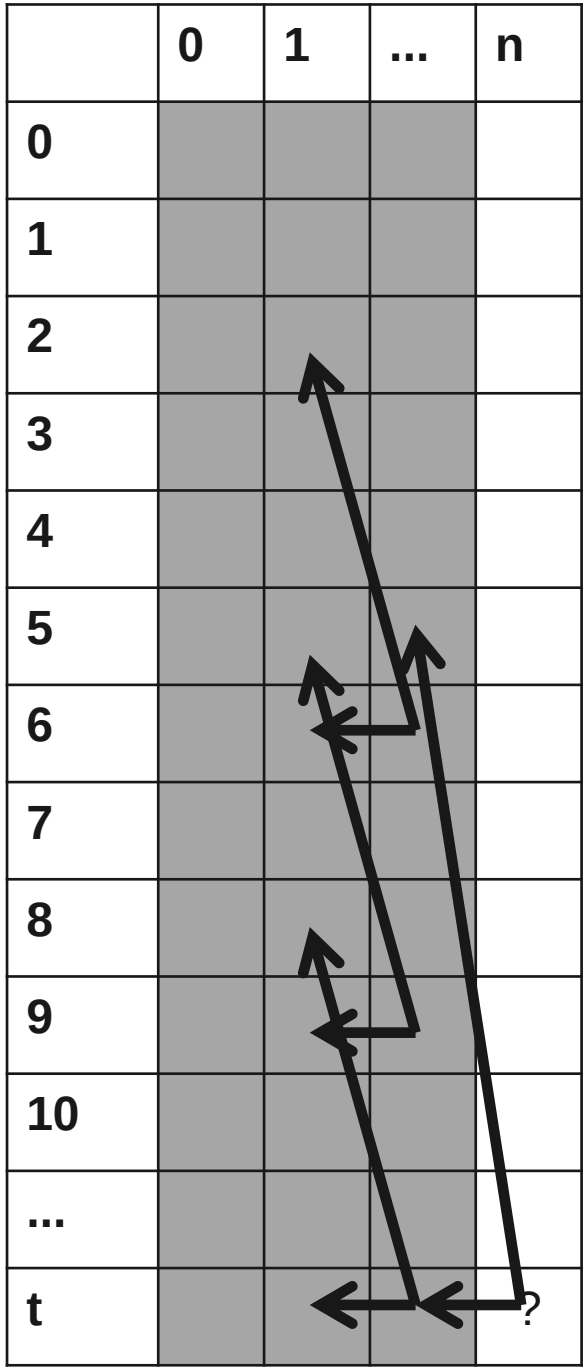

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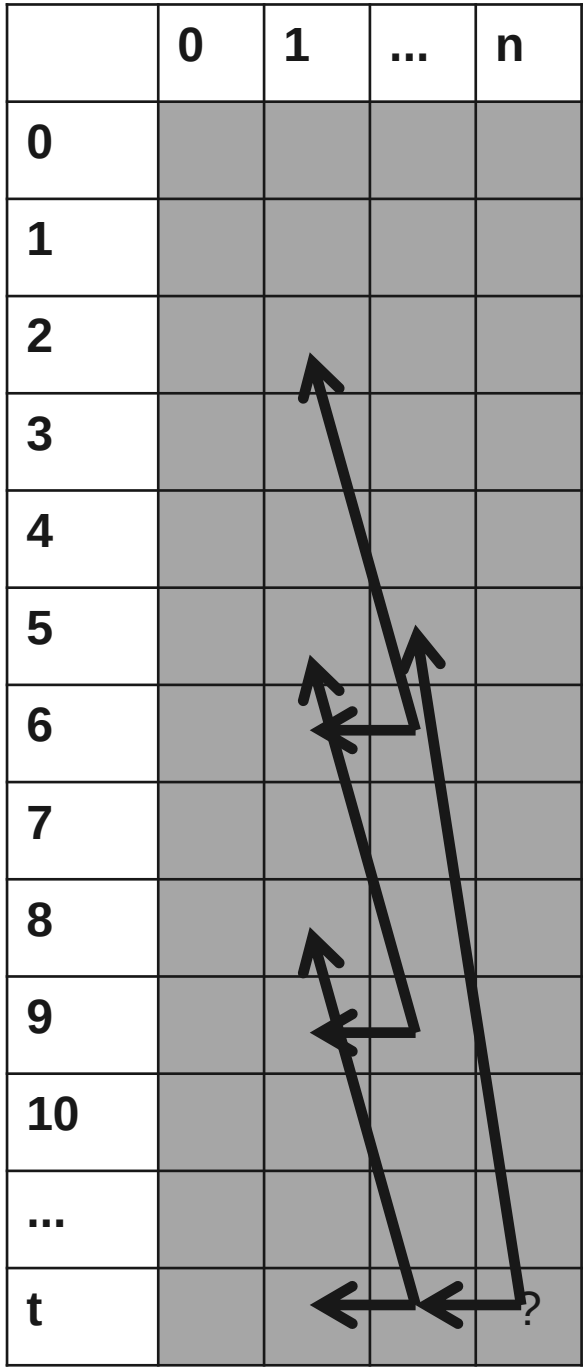

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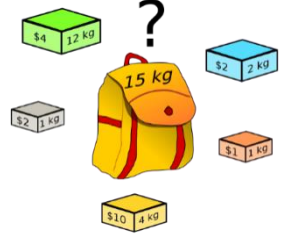
- $O(nt)$ time
- Not polynomial: t can be exponentially large since it is represented in binary!

Knapsack



- Given $w_1, \dots, w_n, b, v_1, \dots, v_n$,t,
  find $X \subseteq \{1, \dots, n\}$ s.t. $\sum_{e \in X} v_e \geq t,\ \sum_{e \in X} w_e \leq b$.
- For $i = 0, \dots, n$ and $j = 0, \dots, V := \sum_e v_e$ let

$$A[i,j] = \min\{\textstyle\sum_{e \in X} w_e : X \subseteq \{1, \dots, i\} \wedge \sum_{e \in X} v_e \geq j\}$$

- $A[i,j]$ is minimum weight of a subset realizing a given value

$$A[i,j] = \begin{cases} 0 & \text{if } i = 0, j \leq 0, \\ \infty & \text{if } i = 0, j > 0, \\ \min\{A[i-1,j], A[i-1, \max\{j - v_i, 0\}]\} & \text{if } i > 0. \end{cases}$$

- Can *find* in $O(nV)$ time set X maximizing $\sum_{e \in X} v_e$ s.t. $\sum_{e \in X} w_e \leq b$:
  - Let v be largest value s.t. $T[n,v] \leq b$
  - Backtrack from $T[n,v]$ (Exercise 6.1)

Approximation scheme for Knapsack

- This DP is again pseudopolynomial

- We'll see a $O(n^3/\epsilon)$ time $(1-\epsilon)$-approximation for the maximization version of knapsack
- I.e., find in time $O(n^3/\epsilon)$ a set A such that:
  - $\sum_{i \in A} w_i \leq b$, and
  - for every X satisfying $\sum_{i \in X} w_i \leq b$
    - $\sum_{i \in A} v_i \geq (1-\epsilon) \sum_{i \in X} v_i$

Approximation scheme for Knapsack

- Idea:
  1. Can replace instance $w_1, \dots, w_n, b, v_1 \cdot S, \dots, v_n \cdot S$, with $w_1, \dots, w_n, b, v_1, \dots, v_n$
     - Optimum stays the same! Useful if S large!
  2. Round values to multiples of S

Approximation for Knapsack

- L4 runs in time

$$O\left(n\sum_i \frac{\tilde{v_i}}{S}\right) = O\left(n\sum_i \frac{v_i}{S}\right) = O\left(n^2\left(\frac{v_{max}}{S}\right)\right) = O\left(\frac{n^3}{\epsilon}\right)$$

- Output A satisfies $\sum_{e\in A} w_e \leq b$

- It remains to compare A with X that reaches OPT

Algorithm `apxKnapsack`$(w_1, \ldots, w_n, v_1, \ldots, v_n, b, \epsilon)$ — assumes $\max_i w_i \leq b$

Output: A s.t. $\sum_{e\in A} w_e \leq b \wedge \sum_{e\in A} v_e \geq (1-\epsilon)OPT$ — $OPT = \max\{\sum_{e\in X} v_e : \sum_{e\in X} w_e \leq b\}$

1: let $v_{\max} = \max_i v_i$, let $S = \lfloor \frac{\epsilon \cdot v_{\max}}{n} \rfloor$
2: **for all** $i = 1, \ldots, n$ **do**
3: $\quad \tilde{v_i} = S \lfloor \frac{v_i}{S} \rfloor$
4: **return** `optKnapsack`$(w_1, \ldots, w_n, \tilde{v_1}/S, \ldots, \tilde{v_n}/S, b)$

Approximation for Knapsack

- Suppose that X is optimal solution (value is $\Sigma_{i \in X}\, v_i$)
- In rounded instance, value of X is
  $\Sigma_{i \in X}\, S\lfloor v_i/S \rfloor \geq \Sigma_{\mathrm{i} \in X}\, \mathrm{v_i} - \mathrm{S}$
- Thus, loss is at most $nS \leq n\frac{\epsilon \cdot v_{max}}{n} \leq \epsilon \cdot v_{max} \leq \epsilon \cdot OPT$ $(OPT \geq v_{max})$

- Value of rounded instance is $\geq OPT - \epsilon \cdot OPT$

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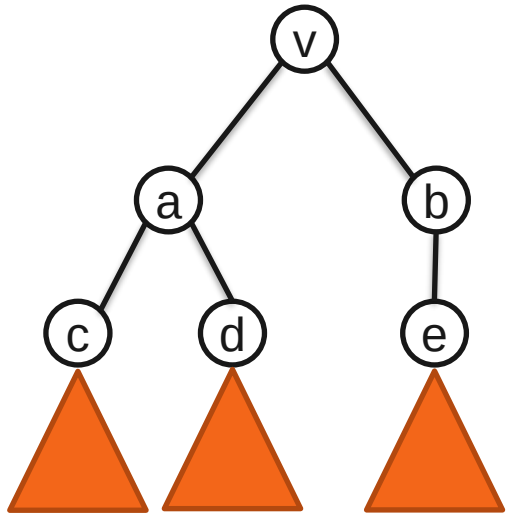
Weighted Independent Set in Trees

- Many NP-hard problem are easy on trees
- In weighted independent set the vertices have weights and we want to find a max. weight ind. set.



- Let T be a rooted tree, $T[v]$ the subtree rooted at v $\in V$. Define $A[v]$ as the max weight of an ind. set of $T[v]$, then

$$A[v] = \begin{cases} \omega(v), & \text{if } T[v] \text{ is a single node,} \\ \max\left\{ \sum_{c \in ch(v)} A[c], \omega(v) + \sum_{c_1 \in ch(v)} \sum_{c_2 \in ch(c_1)} A[c_2] \right\}, & \text{otherwise.} \end{cases}$$

- And we can compute $A[root]$ in $O(n)$ time.

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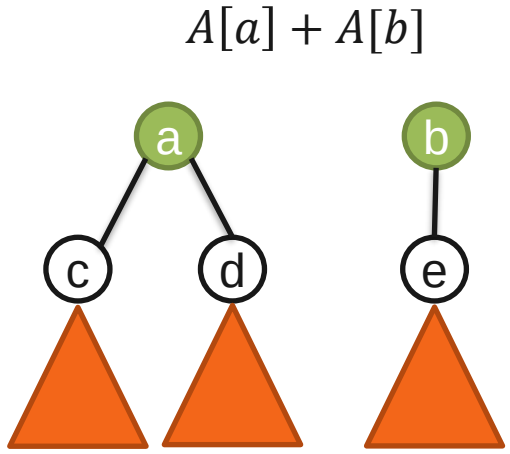

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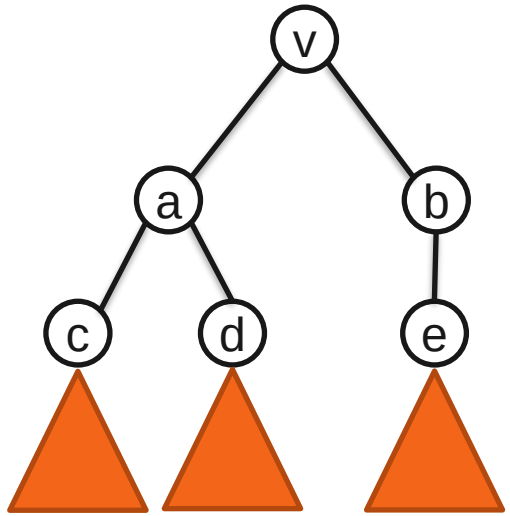

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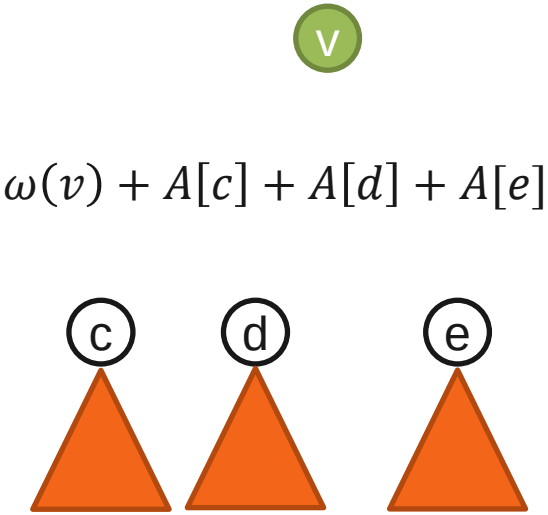

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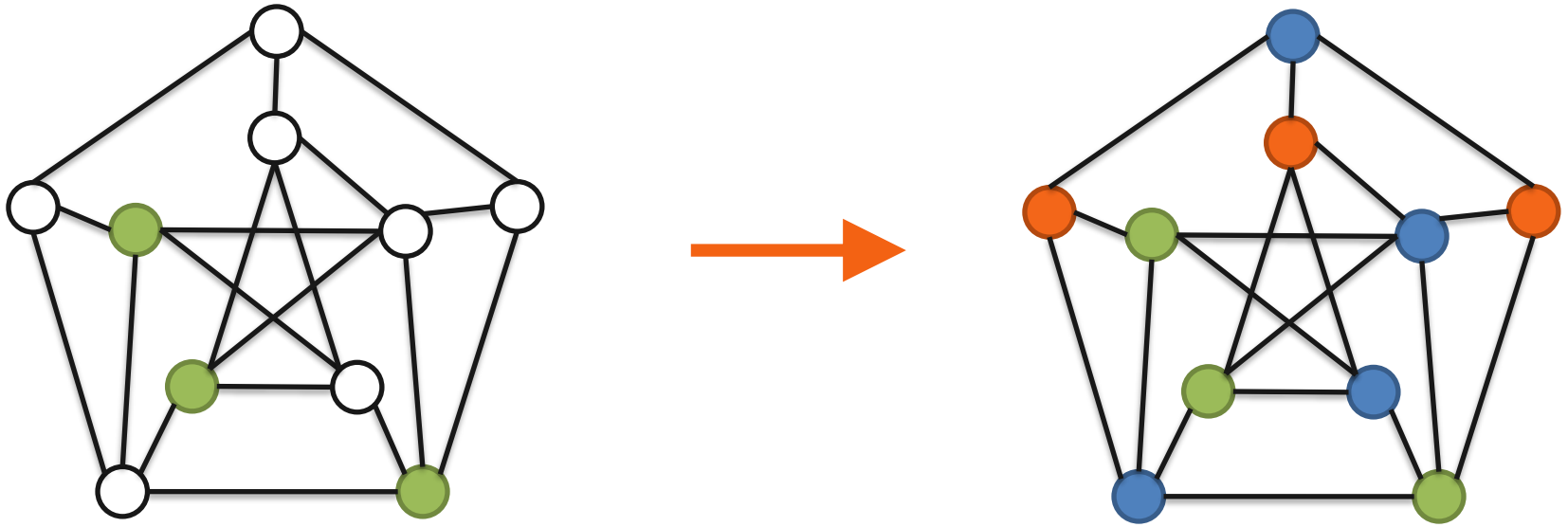
k-coloring in $O^*(3^n)$

- Previous lecture: saw brute force for 3-coloring in $O^*(3^n)$ time; improved to $O^*(2^n)$ time using smarter branching
- Now: k-coloring for arbitrary k in $O^*(3^n)$

k-coloring in $O^*(3^n)$

- Define $A_k[X]$ = [$G[X]$ is k-colorable]

$$A_k[X] = \begin{cases} \textbf{true}, & \text{if } k = 1,\ X \text{ is an independent set of } G \\ \textbf{false}, & \text{if } k = 1,\ X \text{ is not an independent set of } G \\ \bigvee_{Y \subseteq X} A_{k-1}[X \setminus Y] \wedge A_1[Y], & \text{otherwise.} \end{cases}$$



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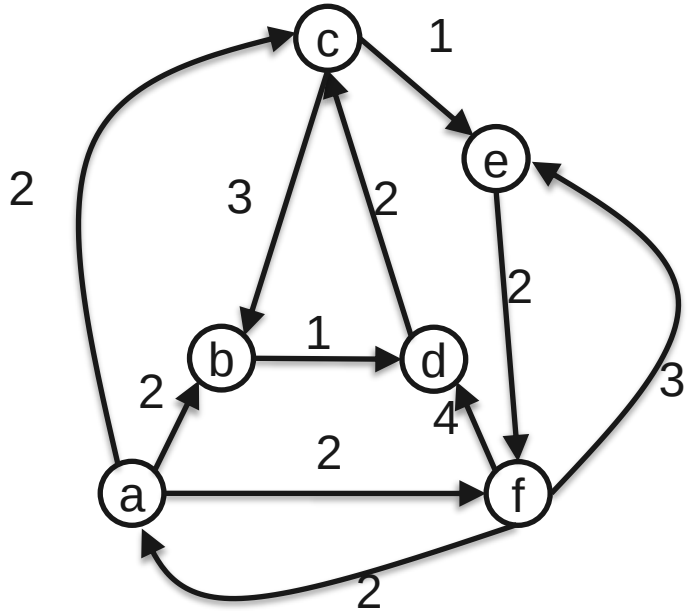
Algorithm `kcolor`$(G = (V, E), k)$
Output: Whether G is k-colorable.
1: **for** $X \subseteq V$ **do**
2: **if** X is an independent set of G **then** set $A_1[X] =$ **true else** set $A_1[X] =$ **false**
3: **for** $l = 2$ to k **do**
4: **for** X
5: set
6: **for**
7: **if**
8: **return**

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k\, y^{n-k}$$

- $poly(n) \sum_{i=1}^{n} \binom{n}{i} 2^i = O^*(3^n)$ time by bin. thm.

Traveling Salesman Problem

- Notation:
  - $\omega: E \to \mathbb{N}$, for $X \subseteq E, \omega(X) = \sum_{e \in X} \omega(e)$.
  - Path P: sequence of distinct consecutively adjacent vertices.
  - $V[P], E[P]$ are used vertices and edges
  - TSP: given $G = (V, E), \omega: E \to \mathbb{N}$. Let $s \in V$.
  - Find path P from s to t with $V[P] = V$ minimizing $\omega(E[P]) + \omega(t, s)$



Traveling Salesman Problem

- Notation:
  - $\omega: E \to \mathbb{N}$, for $X \subseteq E, \omega(X) = \sum_{e \in X} \omega(e)$.
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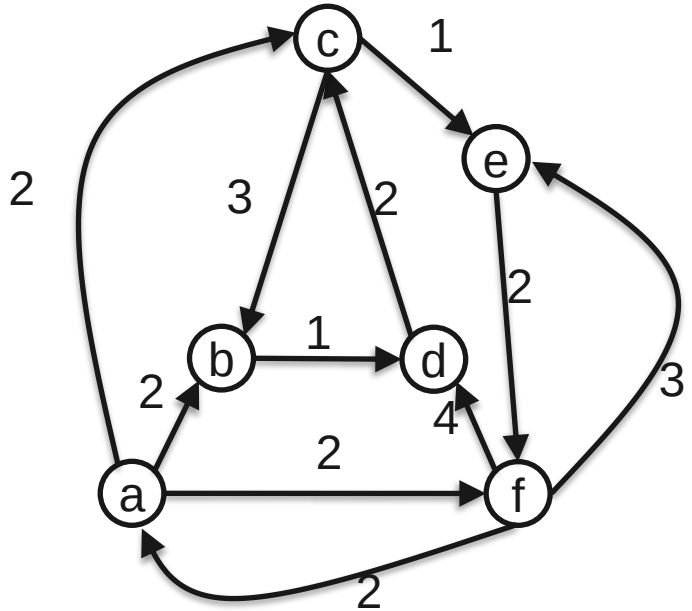
- P=(a,b,d,c)
  - is a path from a to c
  - V[P]={a,b,d,c}
  - E[P] ={ab,bd,dc}
  - $\omega(P) = 2+1+2 = 5$



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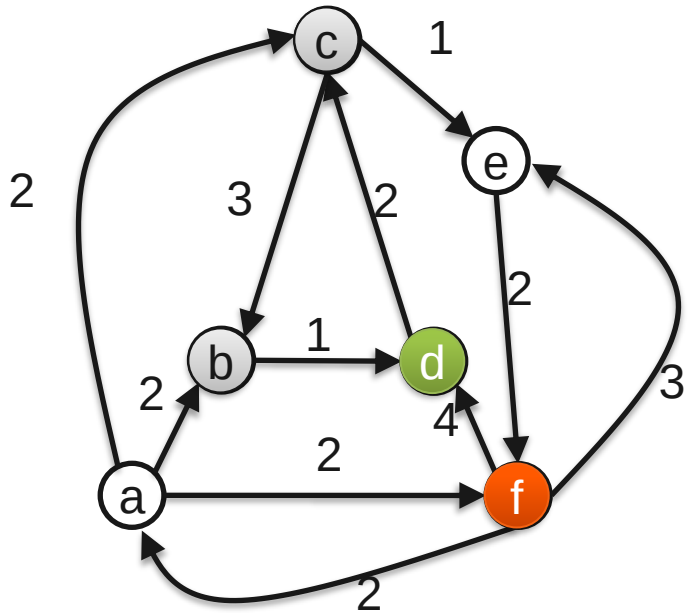
  - only b,f are candidates for t



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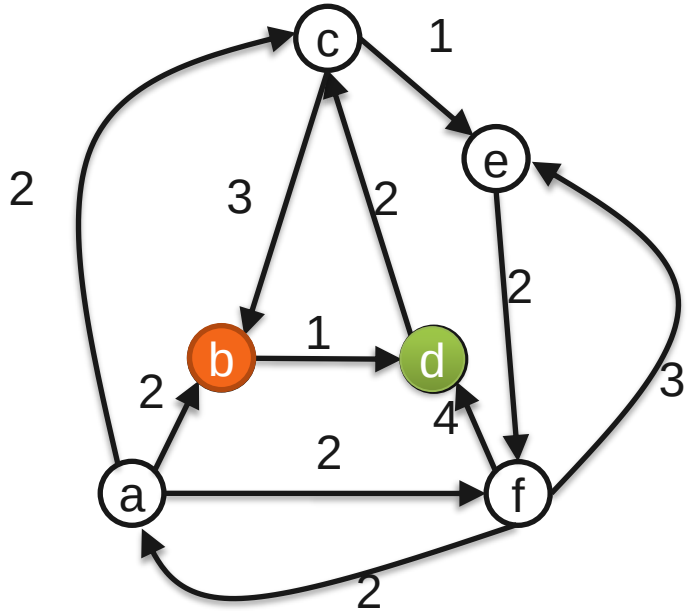
  - P=(d,c,e,f,a,b); $\omega(P) = 9$
  - $\omega(E[P]) + \omega(t, s)$=10



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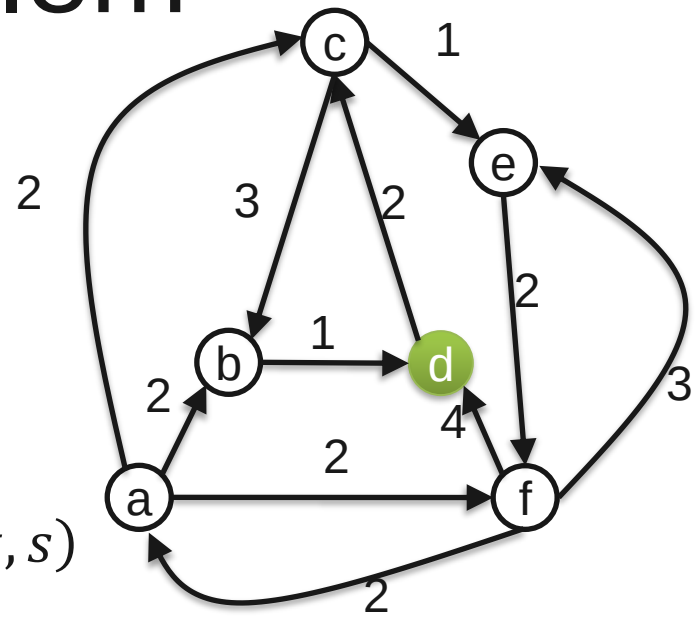


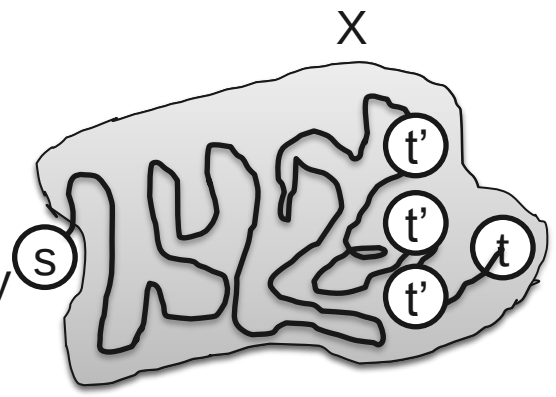
- $A_t[X] = min\{\omega(E[P]): P \text{ path from } s \text{ to } t, \text{ and } V[P] \setminus s = X\}$
  - Only defined for $t \in X, X \subseteq V \setminus s$
  - $A_b[\{c, b\}] = 5$
  - $A_a[\{c, e, f, a\}] = 7$

Traveling Salesman Problem

- $A_t[X] = min\{\omega(E[P]): P \text{ path from } s \text{ to } t, \text{ and } V[P] \setminus s = X\}$

$$A_t[X] = \begin{cases} \infty, & \text{if } |X| = 1 \text{ and } (s,t) \notin E, \\ \omega(s,t), & \text{if } |X| = 1 \text{ and } (s,t) \in E, \\ \min\limits_{t' \in N^-(t) \cap X} A_{t'}[X \setminus t] + \omega(t',t), & \text{otherwise.} \end{cases}$$

- If $|X| = 1$, only allowed path is (s,t)
- If $|X| > 1$:
- To formally argue the equality:
  - Inequalities in both direction:
    - LHS ≥ RHS: every path is built in this way
    - LHS ≤ RHS: adding edge t', t to path obtained from $A_{t'}[X \setminus t]$ gives new path.



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Algorithm `tsp`$(G = (V, E), \omega)$
Output: The minimum $\omega(E[C])$ over all cycles of G (e.g., cycles C satisfying $V[C] = V$).

1: Pick an arbitrary vertex $s \in V$
2: Initiate a table $A_t[X]$ for every $X \subseteq V \setminus s$ and $t \in X$.
3: Set $A_t[\{t\}] = \omega(s,t)$ for every $t \in V$, where $\omega(s,t) = \infty$ if $(s,t) \notin E$.
4: **for** $i = 2$ to n **do**
5: **for** $t \in V \setminus s$ and $X \subseteq V \setminus s$ such that $|X| = i$ and $t \in X$ **do**
6: Set $A_t[X] = \min_{t' \in N^-(t) \cap X} A_{t'}[X \setminus t] + \omega(t',t)$.
7: **return** $\min_{t \in N^-(V)} A_t[X] + \omega(t,s)$.

- Gives $O^*(2^n)$ time.

Inclusion Exclusion

- Classical version: express the size of a union as size of an intersection: if $P_1, P_2 \subseteq U$

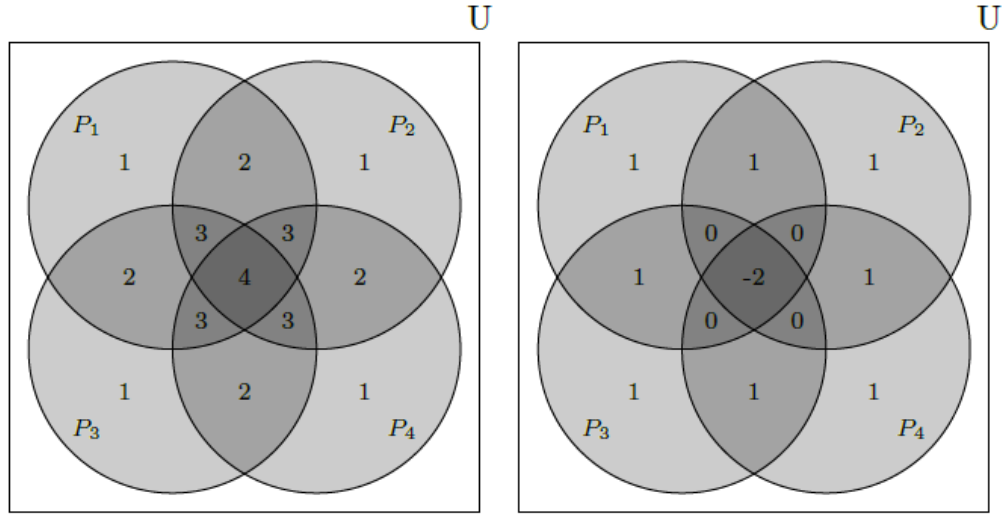
$$|P_1 \cup P_2| = |P_1| + |P_2| - |P_1 \cap P_2|$$

- We are often interested in intersections:

$$|P_1 \cap P_2| = |U| - \left|\overline{P_1}\right| - \left|\overline{P_2}\right| + \left|\overline{P_1 \cap P_2}\right|$$

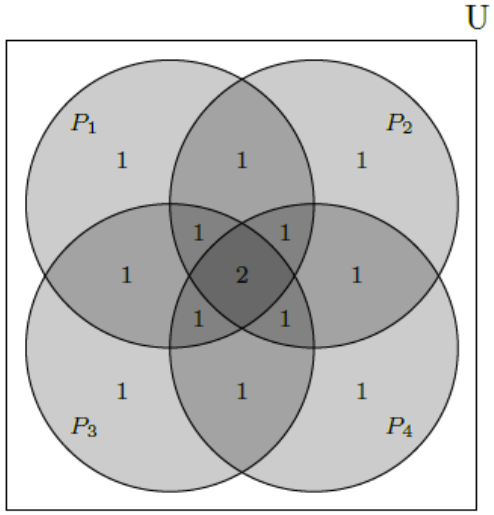
- Generalizes to n sets.

Computing $|P_1 \cup P_2 \cup P_2 \cup P_4|$

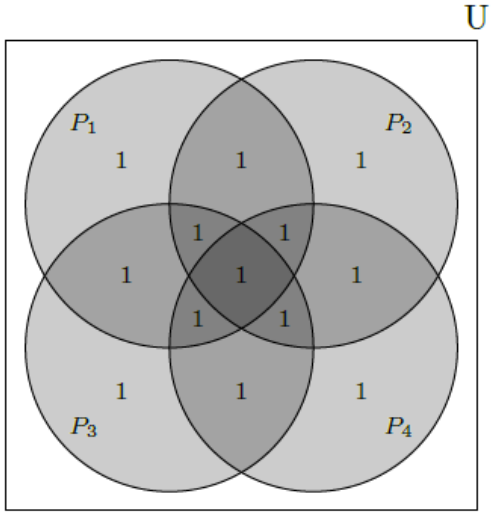


(a) $\sum_i |P_i|$

(b) - $\sum_{i<j} |P_i \cap P_j|$



(c) + $\sum_{i<j<k} |P_i \cap P_j \cap P_k|$



(d) - $\sum_{i<j<k<l} |P_i \cap P_j \cap P_k \cap P_l|$

Inclusion Exclusion Formula

- If $P_1, \dots, P_n \subseteq U, \overline{P_i} = U \setminus P_i, \bigcap_{i \in \emptyset} \overline{P_i} = U$,

$$\left|\bigcap_{i=1}^{n} P_i\right| = \sum_{F \subseteq \{1,\dots,n\}} (-1)^{|F|} \left|\bigcap_{i \in F} \overline{P_i}\right|.$$

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- Proof: for $e \in U$, let $p(e) = \{i : e \in P_i\}$

$$\sum_{F \subseteq \{1,\dots,n\}} (-1)^{|F|} |\bigcap_{i \in F} \overline{P_i}| = \sum_{F \subseteq \{1,\dots,n\}} (-1)^{|F|} |\{e \in U : p(e) \cap F = \emptyset\}|$$

$$= \sum_{e \in U} \sum_{F \subseteq \{1,\dots,n\} \setminus p(e)} (-1)^{|F|}.$$

- $\sum_{Y \subseteq X} (-1)^{|Y|}$ is 1 if $X = \emptyset$ and 0 otherwise by bin thm

$$\sum_{y \subseteq X} (-1)^{|Y|} = \sum_{i=0}^{|X|} \binom{|X|}{i} (-1)^i = (1-1)^{|X|}$$

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- $\sum_{Y \subseteq X} (-1)^{|Y|}$ is 1 if $X = \emptyset$ and 0 otherwise by bin thm (or: every non empty set has equally even as odd subsets)

k-coloring in $O^*(2^n)$ time

- Equivalently: can we cover the vertices with k independent sets?
- Let $U = \{(I_1, \dots, I_k) : I_i \text{ ind. set of } G\}$
- Let $P_v = \{(I_1, \dots, I_k) \in U : v \in I_i, \text{for some i}\}$
- $|\bigcap_{v \in V} P_v|$ =#coverings with k ind. sets

$$|\bigcap_{i=1}^{n} P_i| = \sum_{F \subseteq \{1,\dots,n\}} (-1)^{|F|} |\bigcap_{i \in F} \overline{P_i}|.$$

$|\bigcap_{i \in F} \overline{P_i}|$ =#ind. Sets in $G[V \setminus F]$ to the power k.

k-coloring in $O^*(2^n)$ time

- For $X \subseteq G$, let $i[X] = \#ind.sets\ in\ G[X]$
- Use DP to compute $i[X]$ for every $X \subseteq V$

$$i[X] = \begin{cases} 1 & X = \emptyset \\ i[X \setminus v] + i[X \setminus N[v]] & \text{if } v \in X. \end{cases}$$

- Given this table we can evaluate the formula

$$\sum_{F \subseteq V} (-1)^{|F|} i[V \setminus F]^k$$

- In time $O^*(2^n)$.

Hamiltonian Cycle in $O^*(2^n)$ time and polynomial space

- Notation:
  - *Walk*: sequence P of vertices $p_1, \ldots, p_l$ s.t $(p_i, p_{i+1}) \in E$
  - *Length* of $P = l - 1$ (# of edges)
  - *From s to t*: $p_1 = s$, $p_l = t$, *cyclic* if $p_1 = p_l$
- HC: cyclic walk of length n visiting all vertices.

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- P_v= cyclic walks visiting $v \Rightarrow |\bigcap_{v \in V} P_v| = \#HC$

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$|\bigcap_{i \in F} \overline{P_i}|$ equals #cyclic walks of length n in $G[V \setminus F]$

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  - *From s to t*: $p_1 = s$, $p_l = t$, *cyclic* if $p_1 = p_l$
- $w_F(s, t, k)$ =#length k walks from s to t avoiding F

$$w_F(s,t,k) = \begin{cases} 0 & \text{if } k = 0 \text{ and } s \neq t \\ 1 & \text{if } k = 0 \text{ and } s = t, \\ \sum_{t' \in N^-(t) \setminus F} w_F(s, t', k-1) & \text{otherwise.} \end{cases}$$

- #cyclic walks of length n avoiding F can be computed as $\sum_{s \in V \setminus F} w_F(s, s, n)$