Algorithms and Complexity (AC)

Marie Schmidt & Tom van der Zanden

(Based on slides by Gerhard Woeginger and Jesper Nederlof)

Landelijk Netwerk Mathematische Besliskunde

LNMB, Sep–Nov 2021

Program for today

1. Introduction round
2. Course Intro
3. First lecture

Introduction round

(mentimeter)

(Preliminary) program

13 Sep : Introduction, algorithms, time complexity and computational models, P versus NP
20 Sep : reductions, NP-hardness and NP-completeness
27 Sep : Pseudopolynomial time, strong/weak NP-hardness, co-NP
30 Sept : Approximation algorithms
4 Oct : Exercise set 1
4 Oct : More on approximation algorithms
18 Oct : Exercise set 2
11 Oct : Exact algorithms for NP-hard problems
18 Oct : Exercise set 2
18 Oct : More exact algorithms for NP-hard problems
1 Nov : Exercise set 3
1 Nov : Treewidth
8 Nov : Randomized algorithms
15 Nov : Exercise set 4
15 Nov : **No lecture!!**

Disclaimer: We are discussing famous decision and optimization problems, focusing on their complexity and algorithms to solve them.
We do not, in detail, studying the application of this in real-world planning,

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15 Nov : **No lecture!!**

Website: `https://tomvanderzanden.nl/LNMB-AC/`

First 5 lectures: Marie Schmidt (`schmidt2@rsm.nl`): **life on Zoom**,
last 4 lectures: Tom van der Zanden (`tom.vd.zanden@gmail.com`):

Life lectures

first 5 lectures are going to be life lectures
in some cases: supplemented by videos

lectures are recorded

but:

- please attend
- please turn on your camera
- please participate

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Do we need a 10 minutes break after approx. 45 minutes?

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I do not read the chat while presenting.
If anything goes wrong, please just interrupt me and speak up!
If you have a question: either interrupt me, or wait for the next 'aquamarine' slide.

Attendance

LNMB rule:

- attendance + exercises: 4 EC
- only attendance: 1 EC

Please post '[your name] is here' in the chat for documentation of attendance.

Exercises

1. first series: deadline 4-Oct-2021, 13:15:
   already available on `https://tomvanderzanden.nl/LNMB-AC/`,
   you may want to wait 1 week before solving them
   please email to `schmidt@rsm.nl`
2. second series: deadline 18-Oct-2021, 13:15
   please email to `schmidt@rsm.nl`
3. third series: deadline 1-Nov-2021, 13:15
4. fourth series: deadline 15-Nov-2021, 13:15

May be solved in groups of 2.
My strong advice: do indeed solve them in groups of two!

Questions?

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Program for the first three weeks

- algorithms,
- computational models and (worst-case) time complexity
- decision problems, P versus NP
- Reductions
- NP-hardness
- A catalogue of NP-hard problems
- pseudo-polynomial time
- strong NP-hardness & weak NP-hardness
- co-NP, co-NP versus NP

What is an algorithm?

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Example: Insertion Sort

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for $j = 2, \ldots, n$ **do**
  $key := A[j]$
  $i := j - 1$
  while $i > 0$ and $A[i] > key$ **do**
    $A[i + 1] := A[i]$
    $i := i - 1$
  end while
  $A[i + 1] := key$
end for
return A

When we analyze an algorithm, we are interested in:

- running time of the algorithm
- space (memory) needed by the algorithm
- for optimization problems: quality of the output
  - exact algorithm
  - approximation algorithm
  - heuristic algorithm

Time and space complexity of 'Insertion Sort'

- How much time did we need?
- **How much space did we need?**
  - in our example
  - for sorting any sequence of 4 numbers
  - for sorting any sequence of n numbers

Big-O notation

We make use of big-O notation for discussing encoding space, space complexity, and time complexity.

big-O notation

$f(n)$ is $O(g(n))$ denotes
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  Definition of 'elementary operation' depends on computational model

- How much space did we need?

Time complexity: computational model

Our choice: Random-access-machine (RAM) model

executes operations one after another (no concurrent operations)

Elementary steps $\hat{=}$ assumption: can be executed in constant time

- arithmetic: add, subtract, multiply, divide, remainder, floor, ceiling
- data movement: load, store, copy
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Why do we use RAM:

- similar to how a computer works & approximates running time of computer well

Time Complexity of Insertion Sort

- How many elementary steps did we make to sort the sequence $(5, 2, 4, 1)$ with Insertion sort?
- How many elementary steps do we need, at most, for sorting a sequence of 4 numbers?
- How many elementary steps do we need, at most, for sorting a sequence of n numbers?

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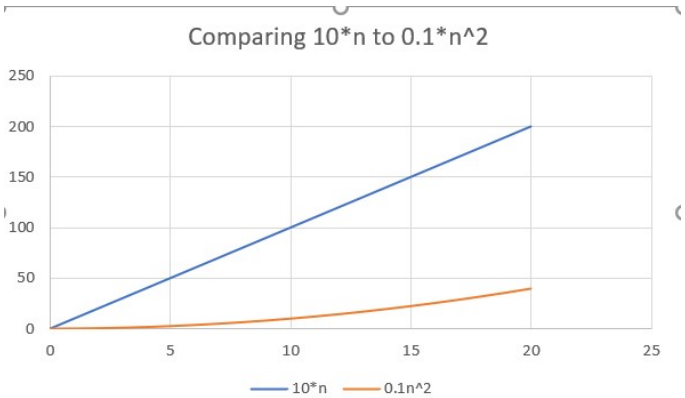
Comparing running times

What is better: an algorithm that solves your problem in time $O(n)$, or one that solves it in time $O(n^2)$?

Comparing running times



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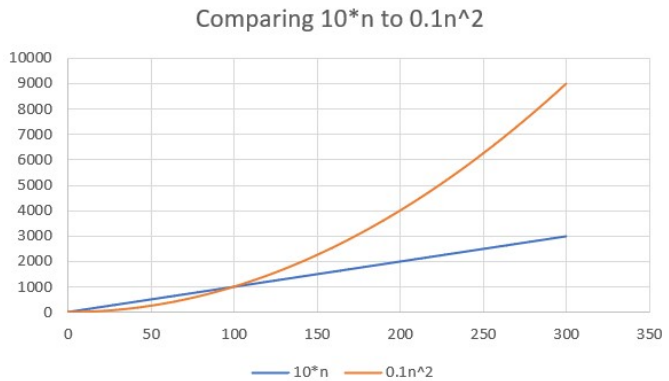

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As we are most worried about solving large problem instances:
We consider $O(n)$ to be better than $O(n^2)$,
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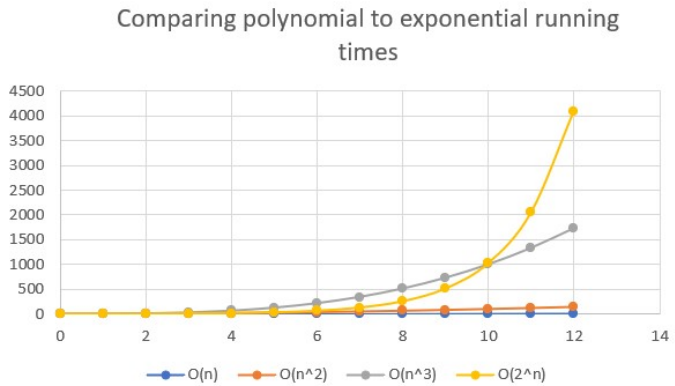

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...
...and any *polynomial-time algorithm* to be better than an *exponential-time algorithm*

Vocabulary

Time complexity of algorithms

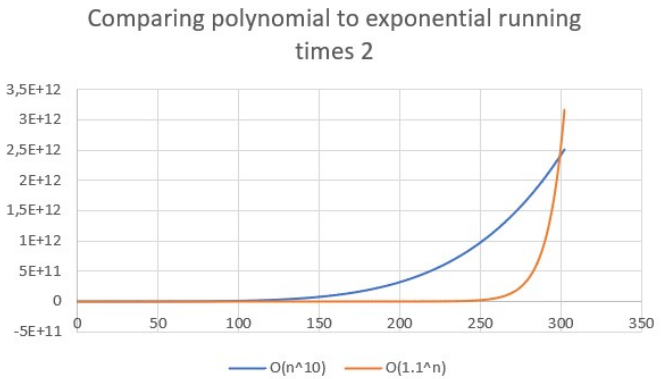
We call an algorithm *polynomial-time algorithm*, if there is a number i such that the algorithm's worst-case running time is in $O(n^i)$.
Otherwise we call it an *exponential-time algorithm*.

Exponential-time algorithms

- Refer to any algorithm whose worst-case running time is not in $O(n^i)$ for a fixed i.

Exponential-time algorithms

- Refer to any algorithm whose worst-case running time is not in $O(n^i)$ for a fixed i.
- Are considered as 'inefficient' by many people **Intuition:**
  Polynomial = desirable, good, harmless, fast, short, small
  Exponential = undesirable, bad, evil, slow, wasteful, horrible
- But: can be better than polynomial-time algorithms for specific problem instances (see also later sessions of this course)!



Other approaches to measure algorithm complexity

- Average case (instead of worst-case) complexity
  (not treated in this course)
- Complexity measured not (only) in input length n
  Alternatives:
  - size of the output (e.g., in multicriteria optimization)
  - numbers contained in program input $\rightarrow$ more on this when we talk about 'pseudo-polynomial' algorithms
  - instance characteristics $\rightarrow$ more on this when we talk about 'fixed-parameter tractability'

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Problems are specified in terms of
problem input: what is given
problem output: what is searched for

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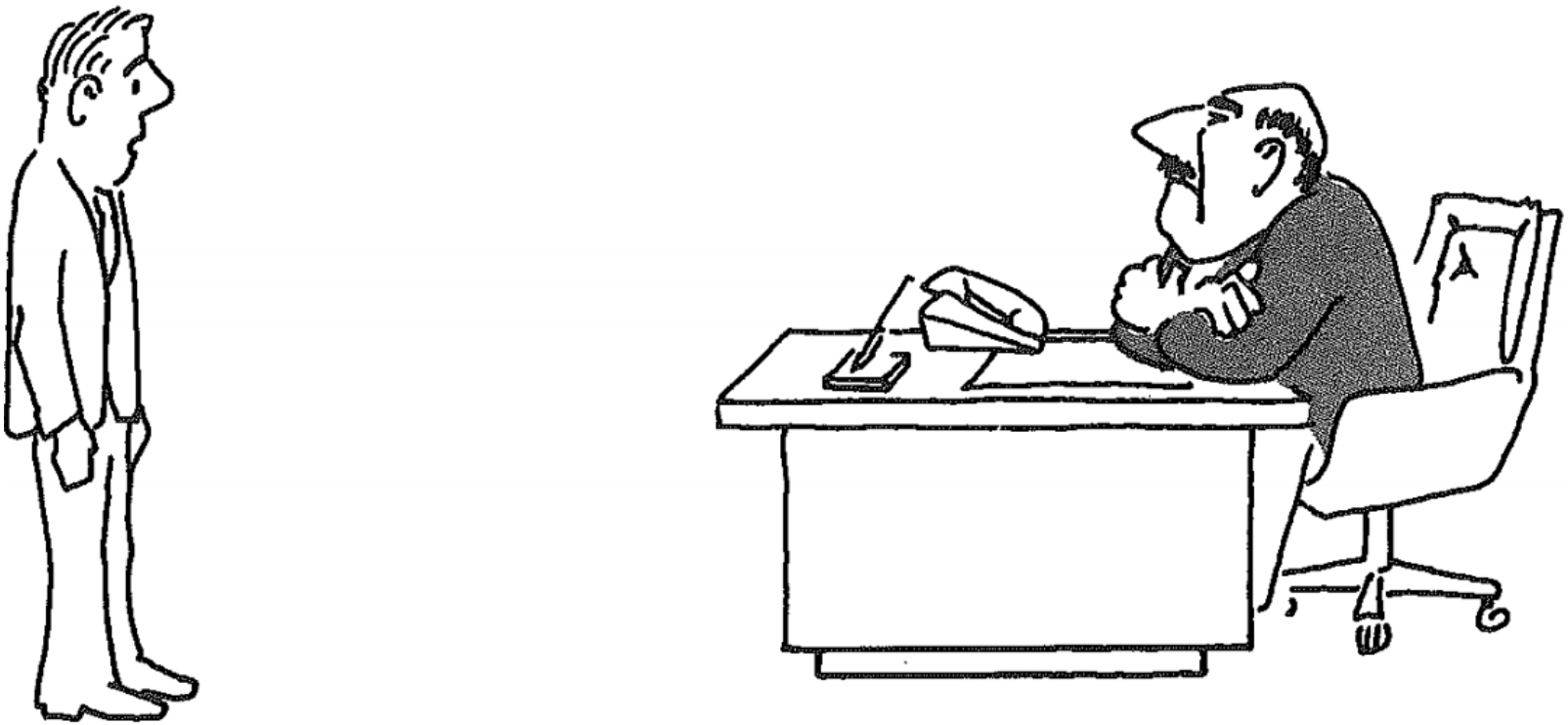
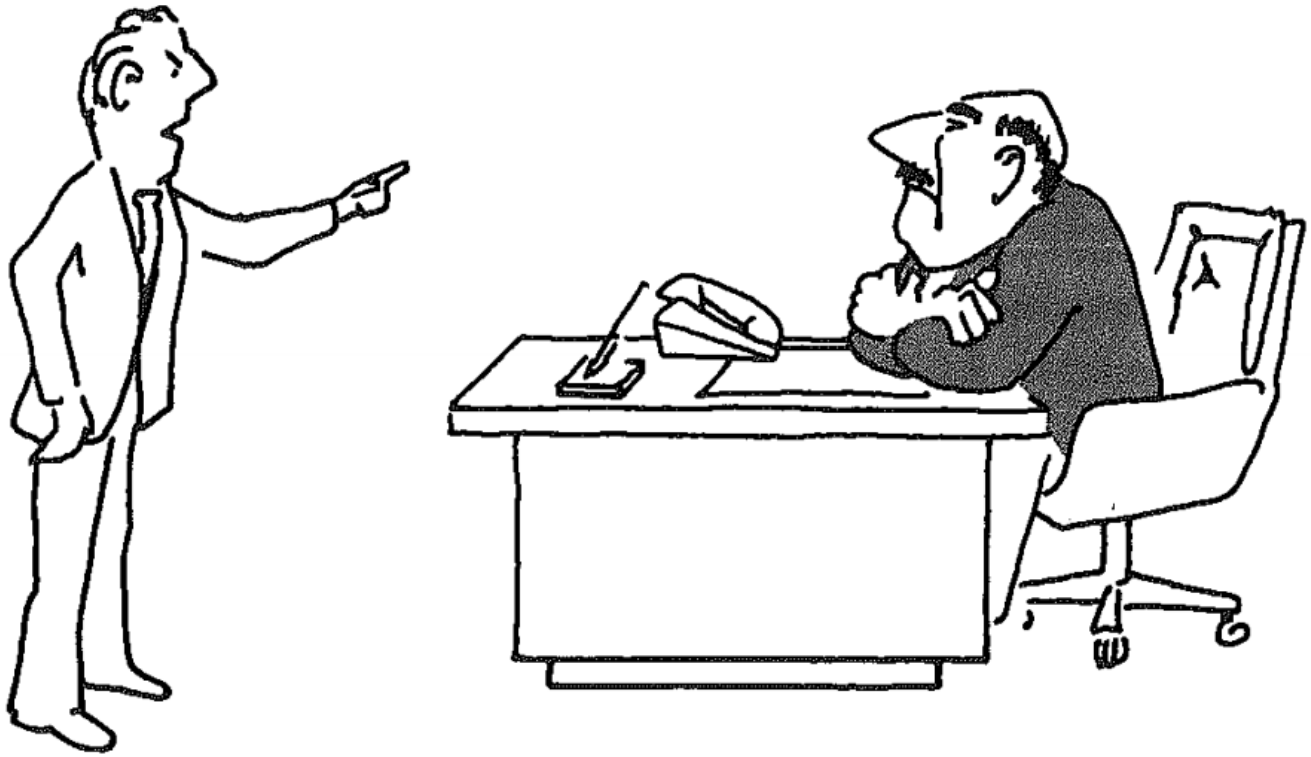
Selection of famous sorting algorithms (see wikipedia for many more):

- Insertion sort ($O(n^2)$)
- Merge sort ($O(n \log(n))$)
- Bubble sort ($O(n^2)$)
- Shell sort ($O(n^{3/2})$)
- Bogosort ($O((n+1)!)$)



"I can't find an efficient algorithm, I guess I'm just too dumb."



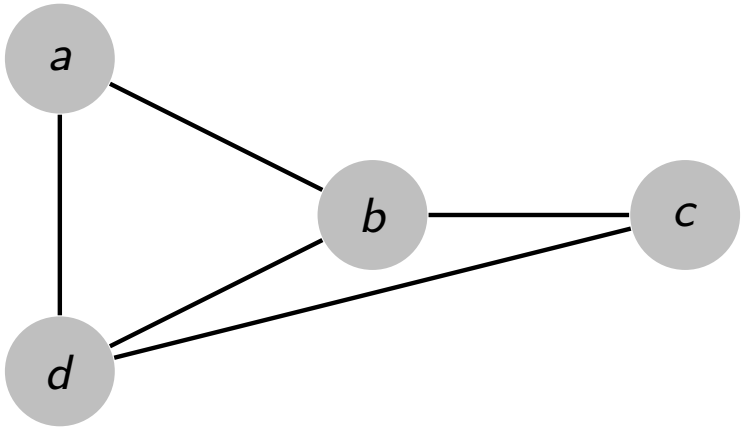
"I can't find an efficient algorithm, because no such algorithm is possible!"

Decision problems and optimization problems

In the remainder of the course, we are mostly concerned with the following two types of algorithmic problems:

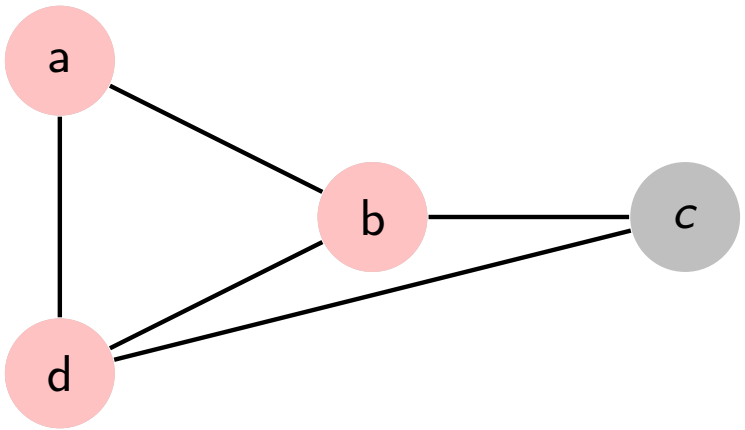
- Optimization problems (min/max)
- Decision problems (with answer YES/NO)

Decision problems and optimization problems



Reminder 'clique': set of pairwise adjacent vertices

Decision problems and optimization problems



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Instance: a graph $G = (V, E)$; a bound k
Question: does G contain a clique of size (at least) k?

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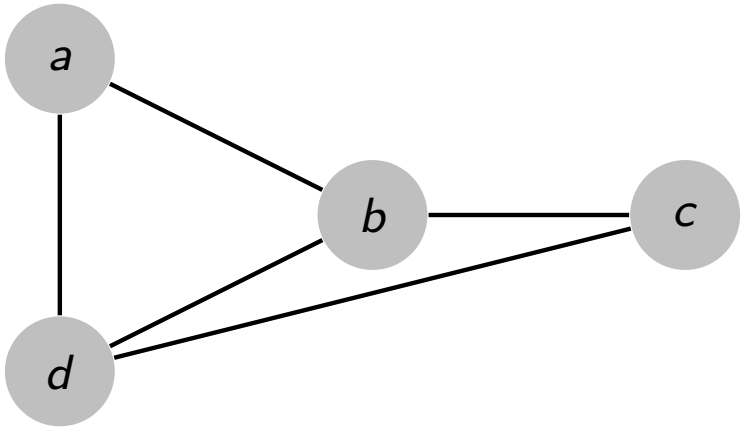

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Example: Optimization problem CLIQUE

Instance: a graph $G = (V, E)$
Goal: find a clique of maximum size in G. / What is the maximum size of a clique in G?

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Every discrete optimization problem can be rewritten into a sequence of decision problems:

use bisection search on the interval of objective values

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Let G be a graph on n vertices.

Does G contain a clique of size at least $n/2$? – YES

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Let G be a graph on n vertices.

Does G contain a clique of size at least $n/2$? – YES
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Let G be a graph on n vertices.

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Complexity class P

Remember: Polynomial growth rate:

- $O(\text{poly}(n))$ for some polynomial poly Example: $O(n)$; $O(n \log n)$; $O(n^3)$; $O(n^{100})$

Exponential growth rate:

- everything that grows faster than polynomial

Example: 2^n; 3^n; $n!$; 2^{2^n}; n^n

Intuition:

Polynomial = desirable, good, harmless, fast, short, small
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If you are interested in what a Turing machine is, and how it relates to algorithms, please watch the (additional) video provided on our course website

Example: The Minimum Spanning Tree (MST) problem

'Minimum Spanning Tree' (MST) - Decision version

Given: a graph $G = (V, E)$ and $w_e \in \mathbb{R}$ for every $e \in E$, a number W
Question: is there a spanning tree $T \subseteq E$ such that $\sum_{e \in T} w_e \leq W$

Terminology:

- tree: edge-set without cycles (e.g. at most 1 path between 2 vertices)
- spanning: all vertices are incident to an edge

Kruskal's algorithm for Minimum Spanning Tree

Computes a Minimum Spanning Tree T using a *greedy* approach:

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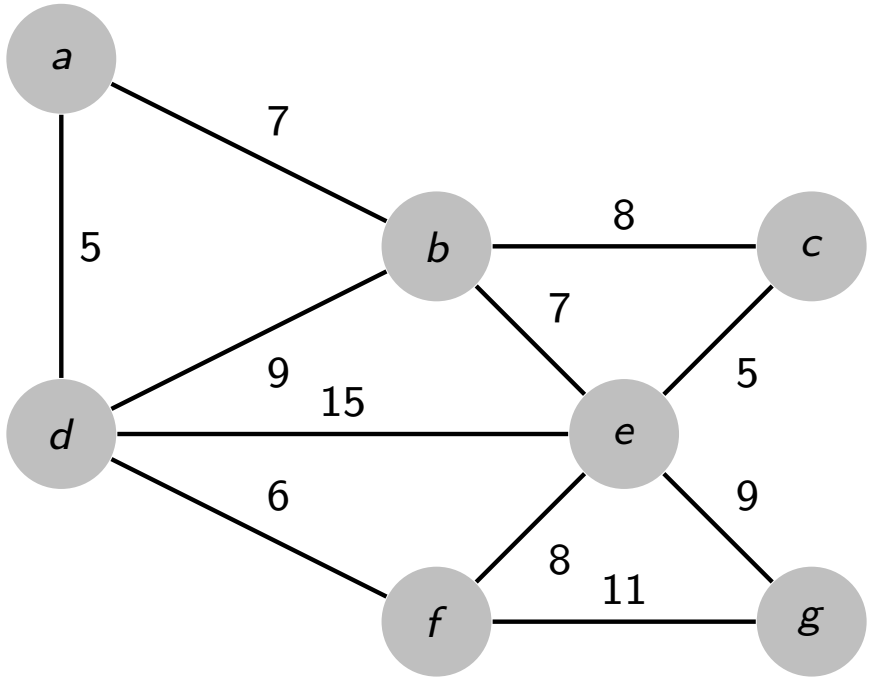

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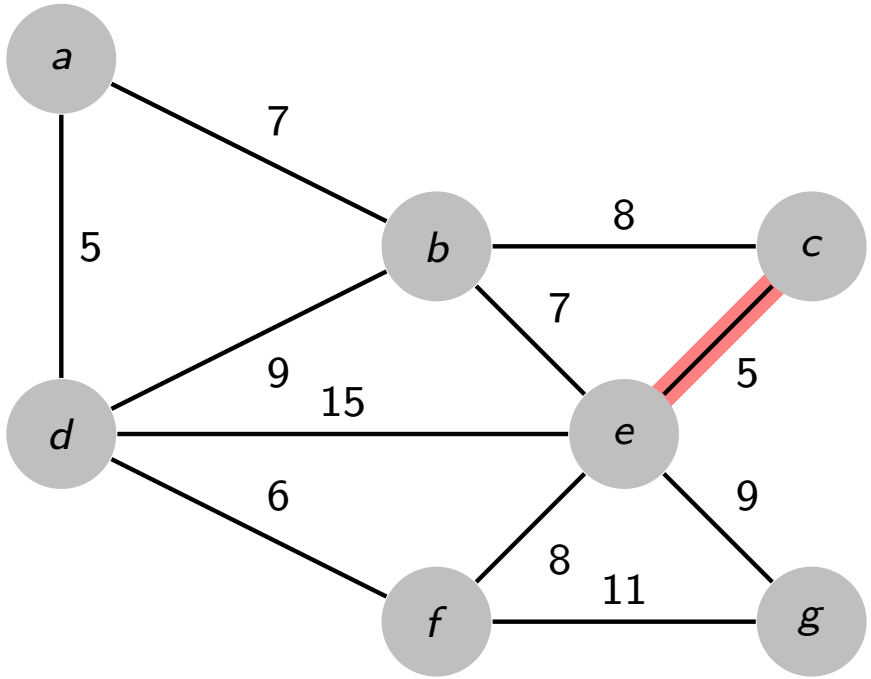

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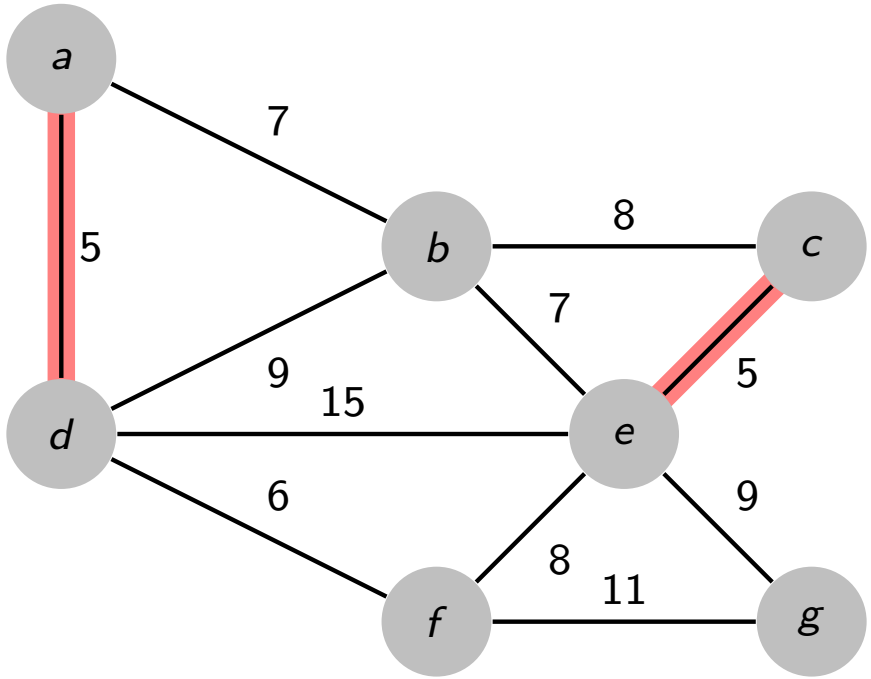

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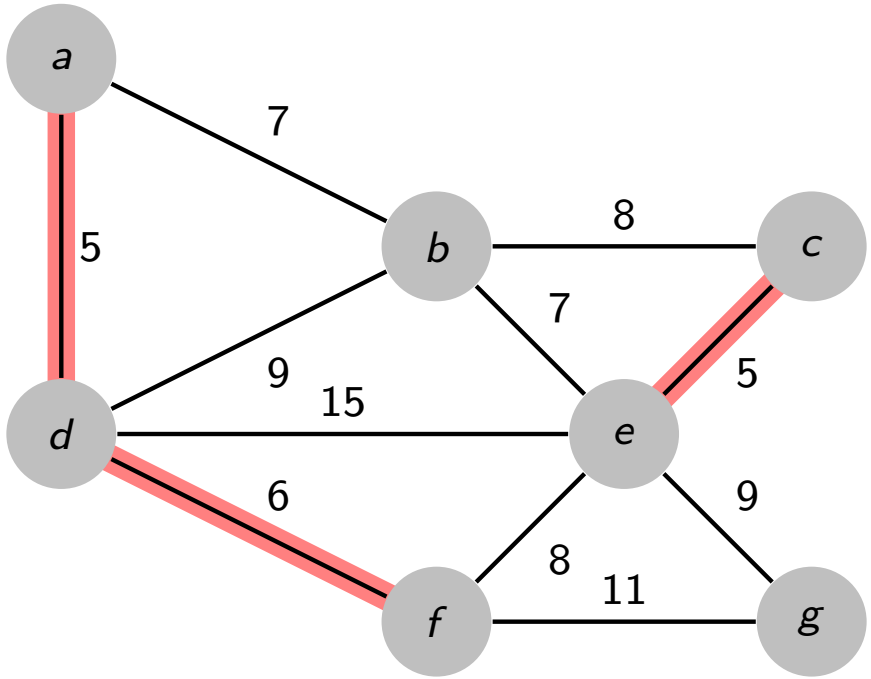

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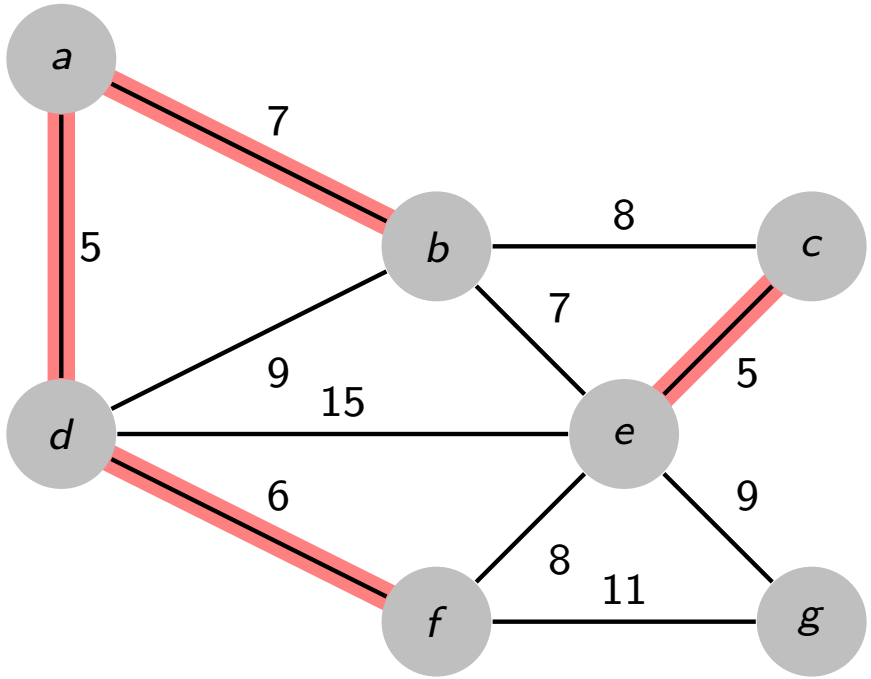

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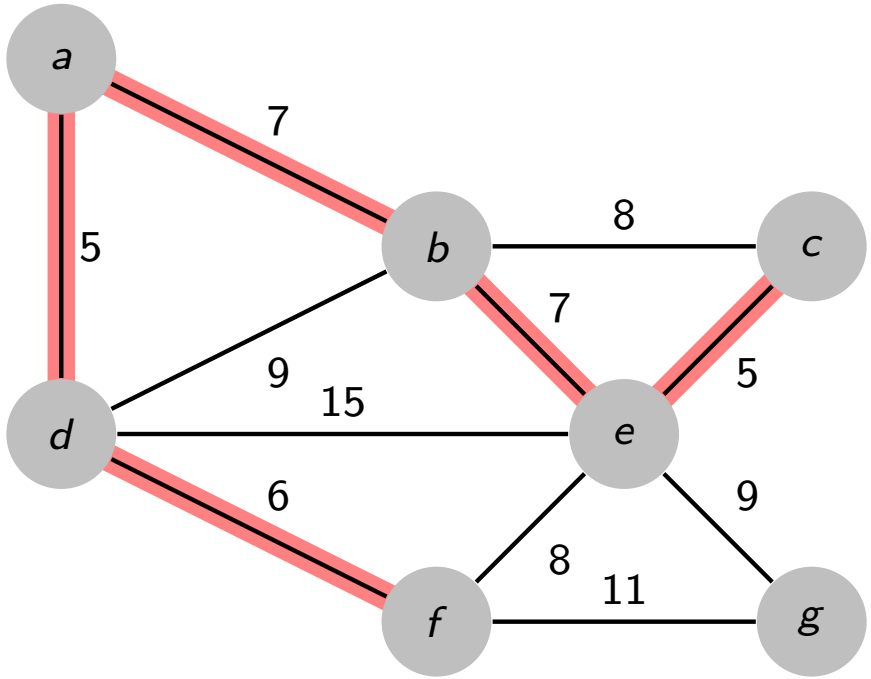

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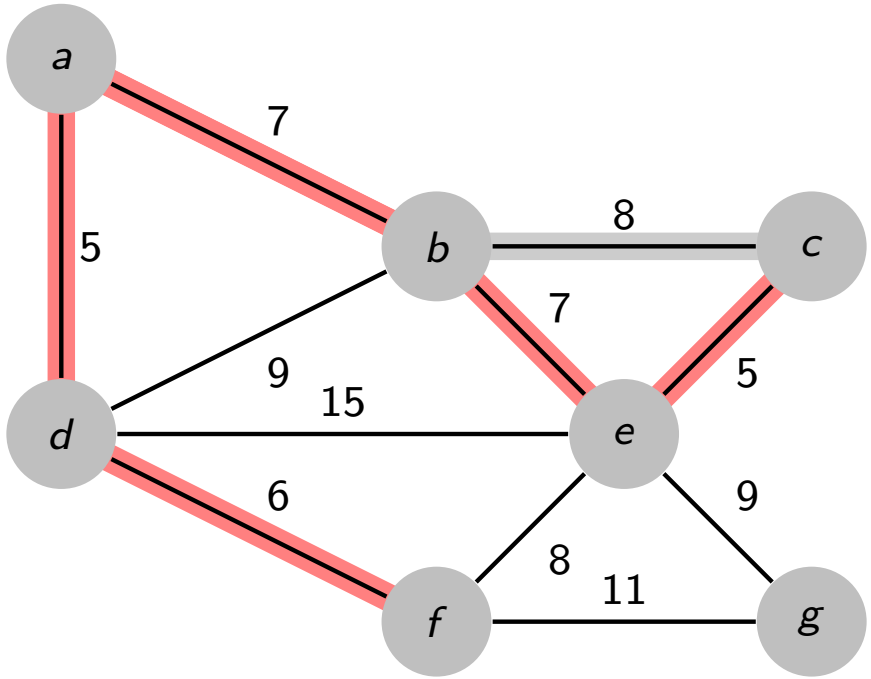

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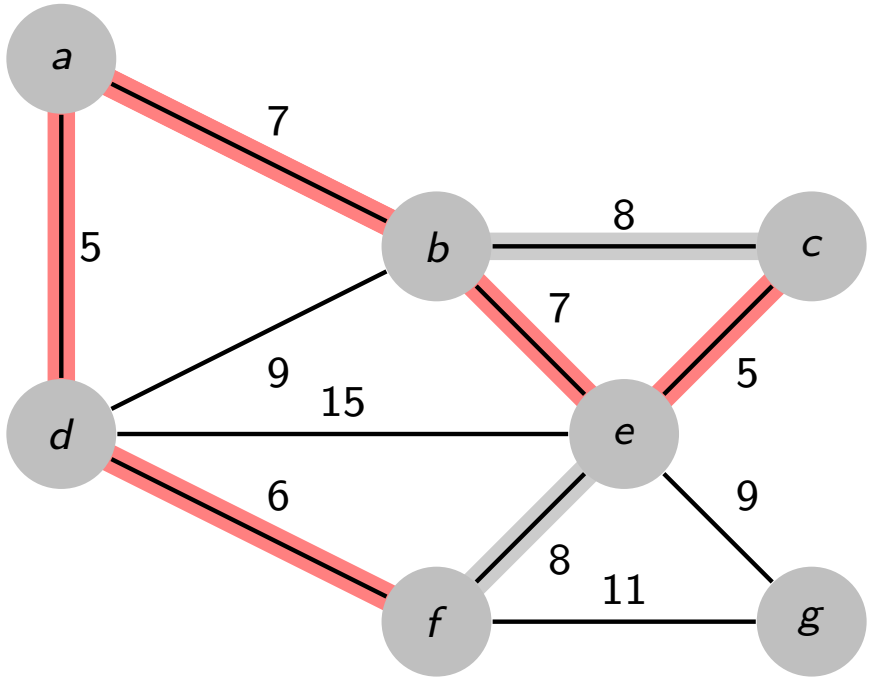

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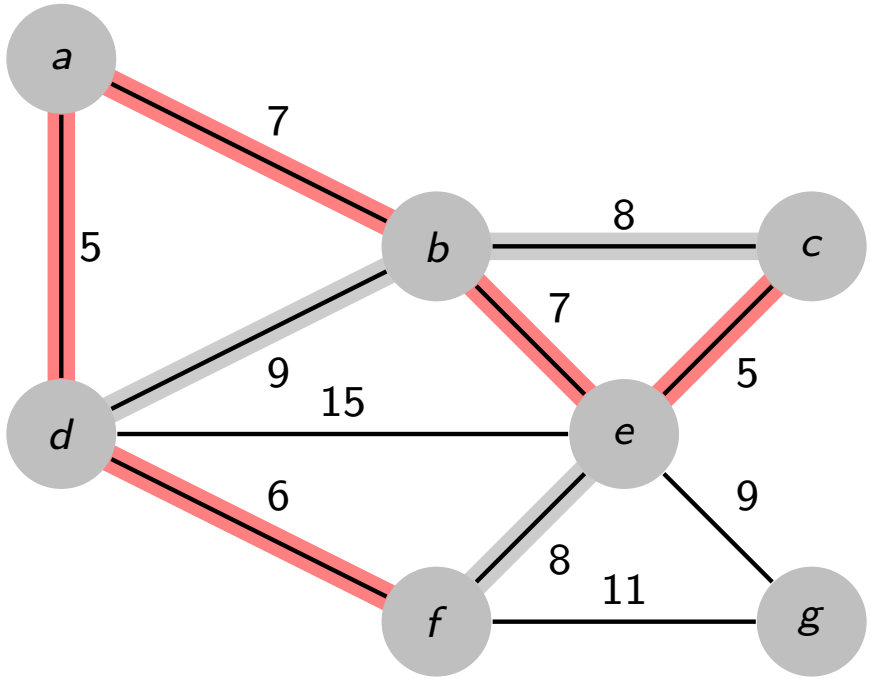

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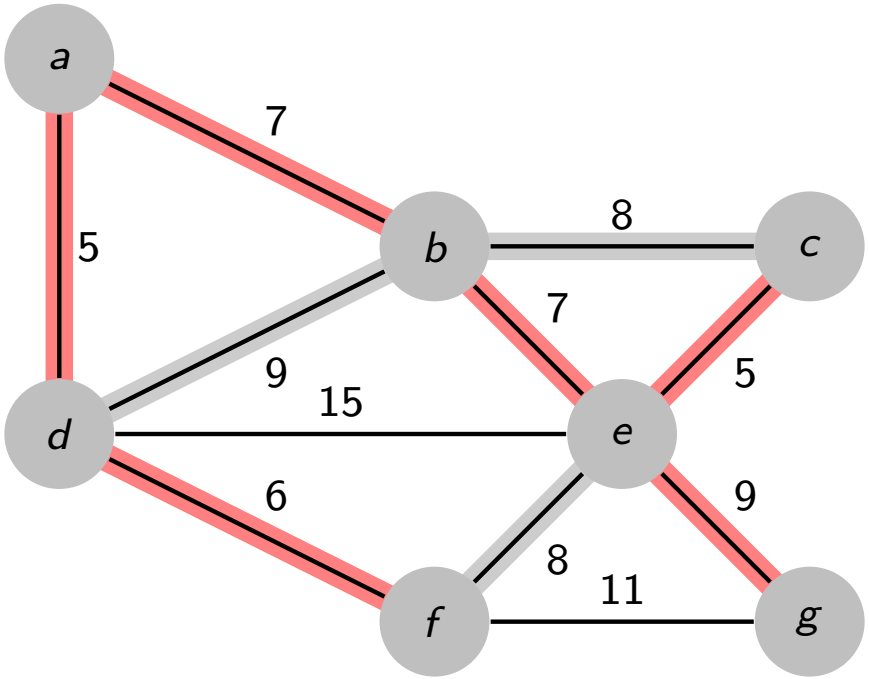

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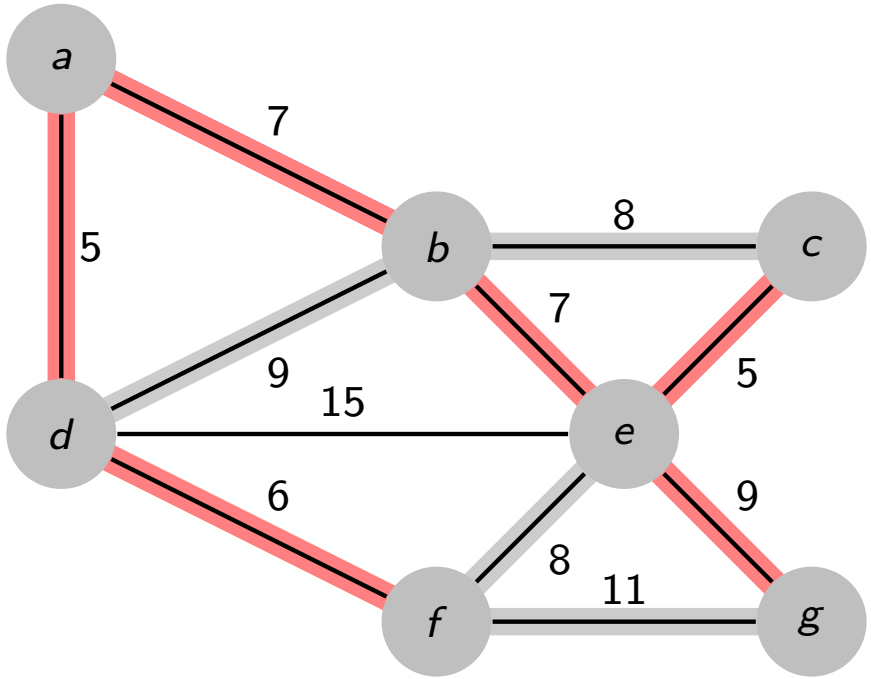

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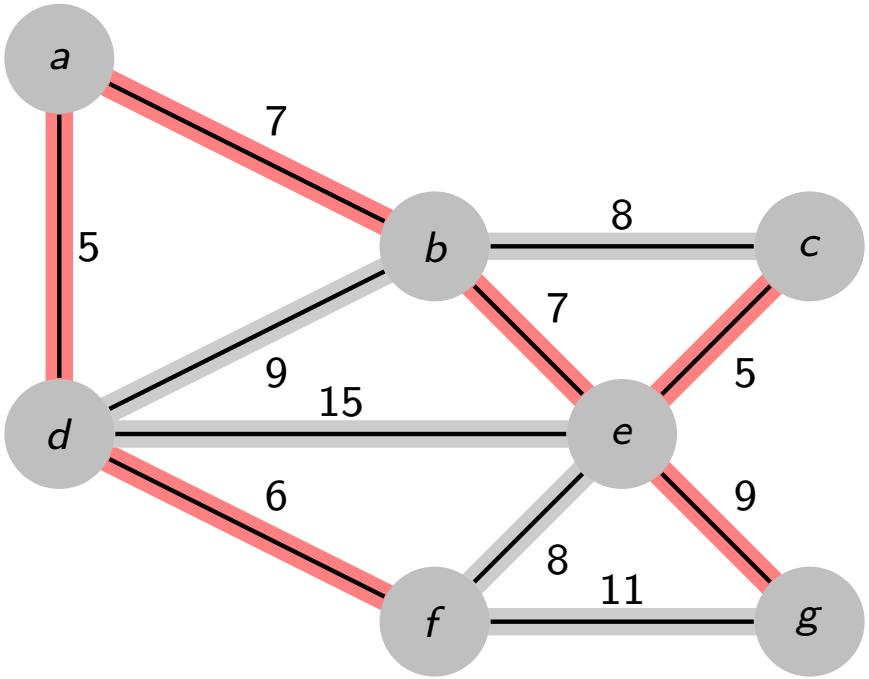

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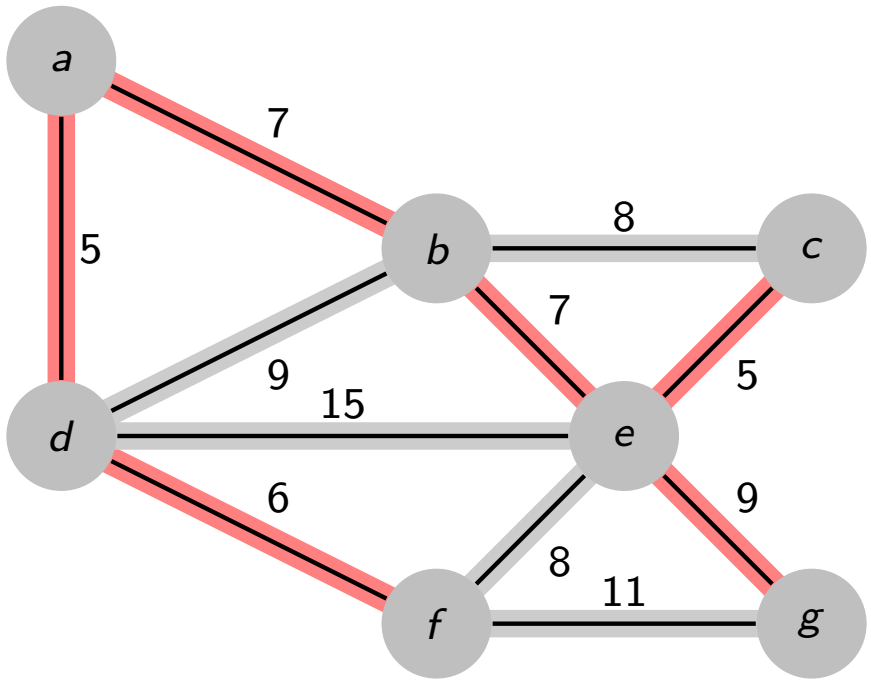

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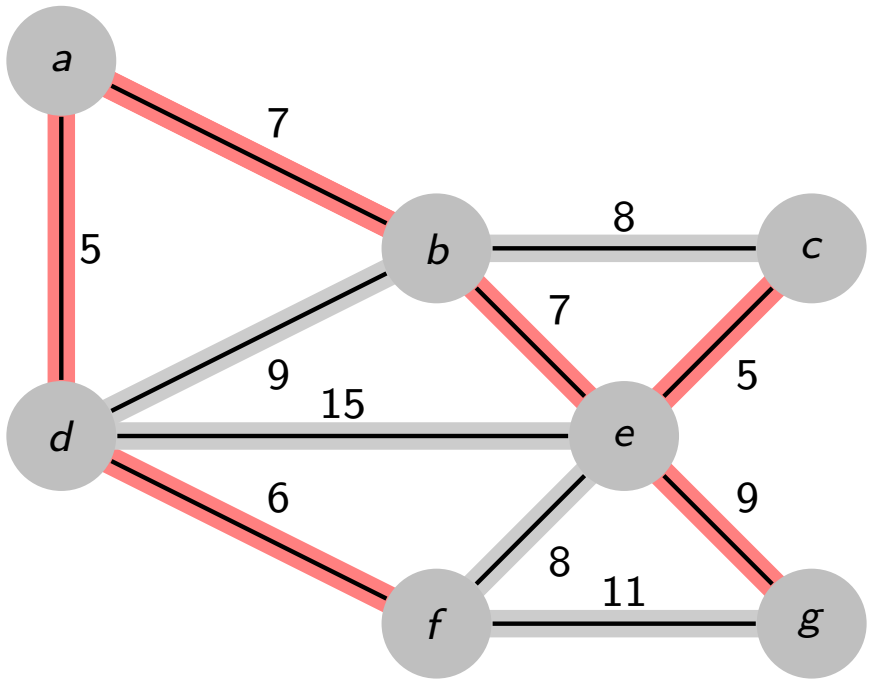

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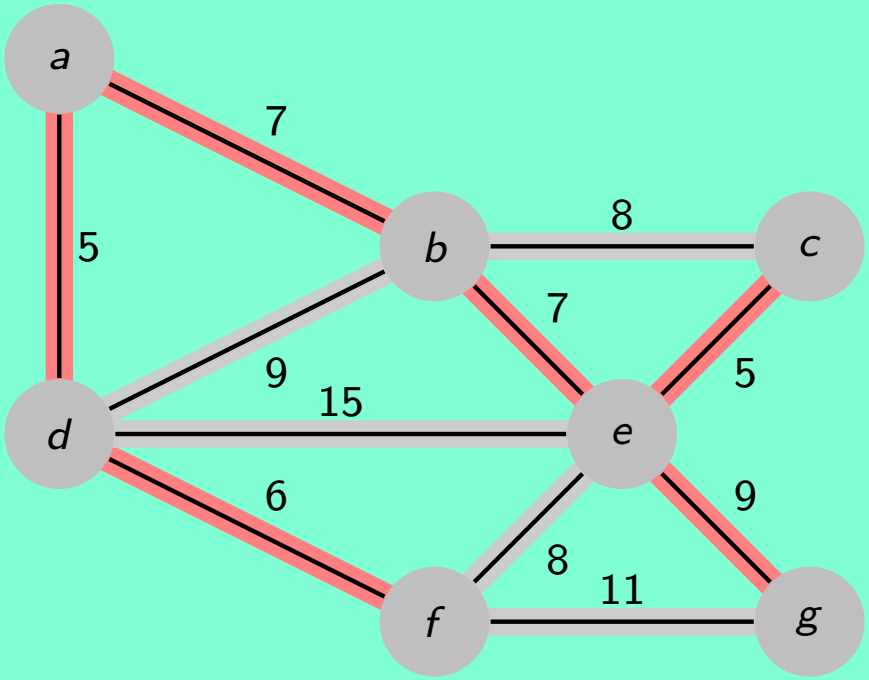

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Instance: cities $1, \ldots, n$; distances $d(i, j)$; a bound B
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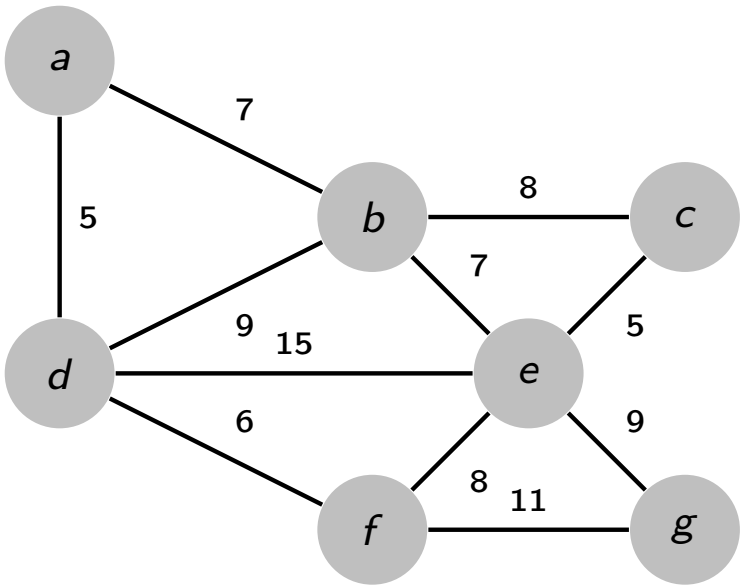

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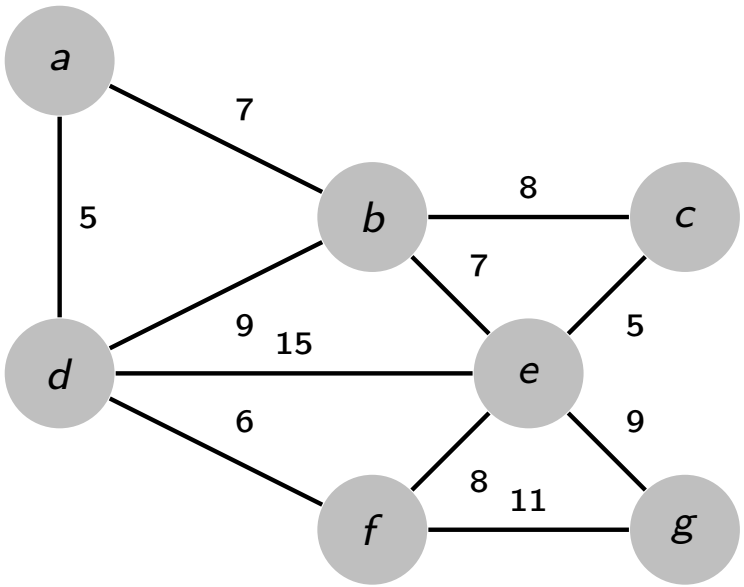

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- Specify sequence of edges.

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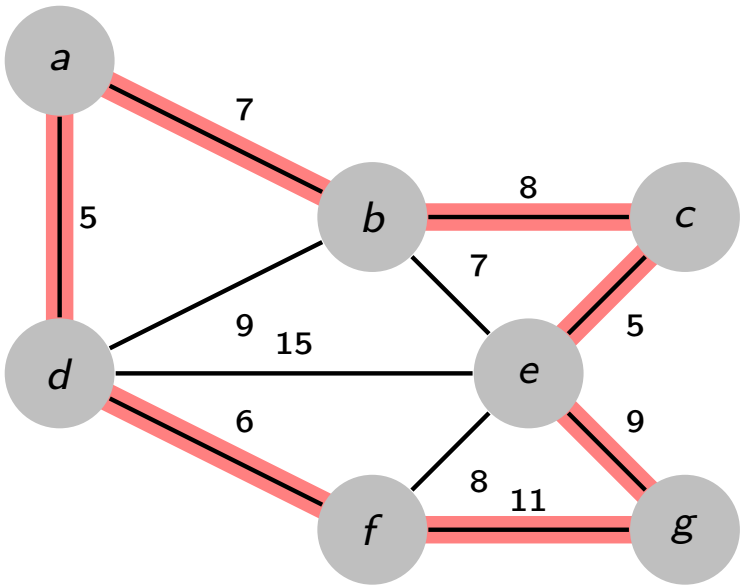

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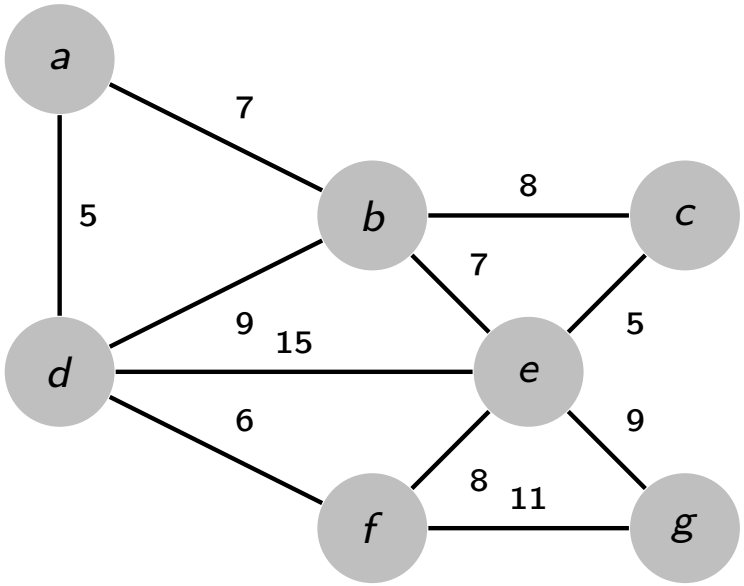

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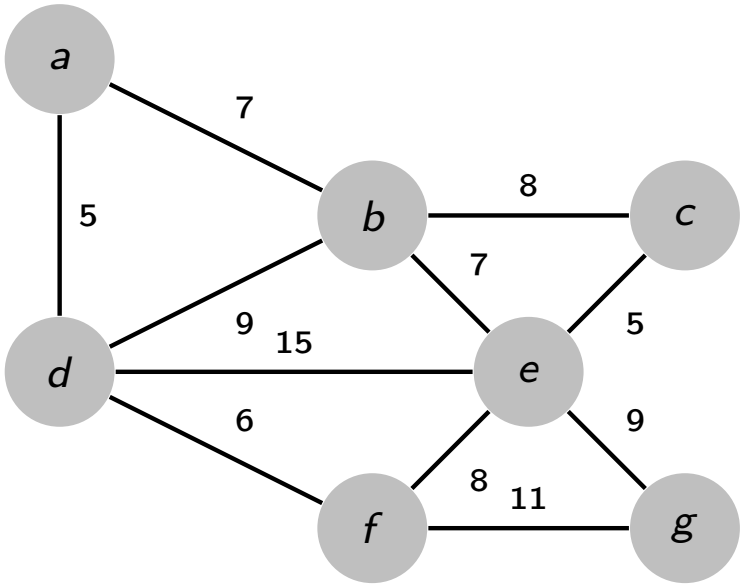

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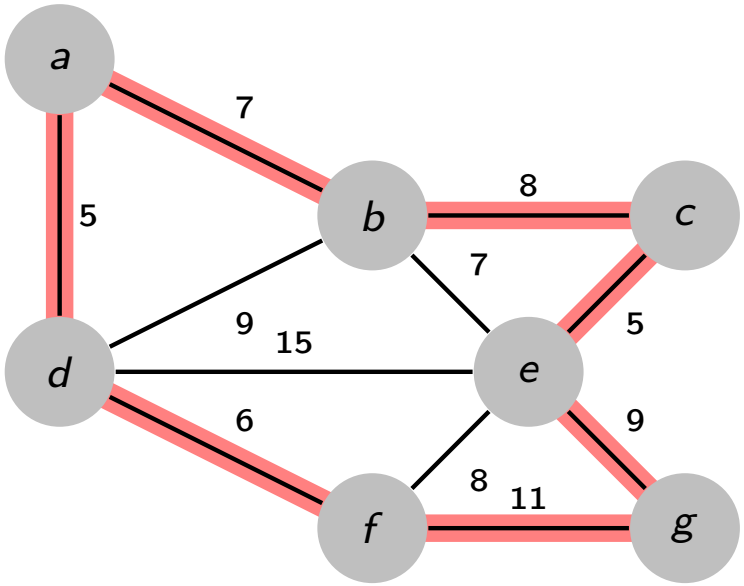

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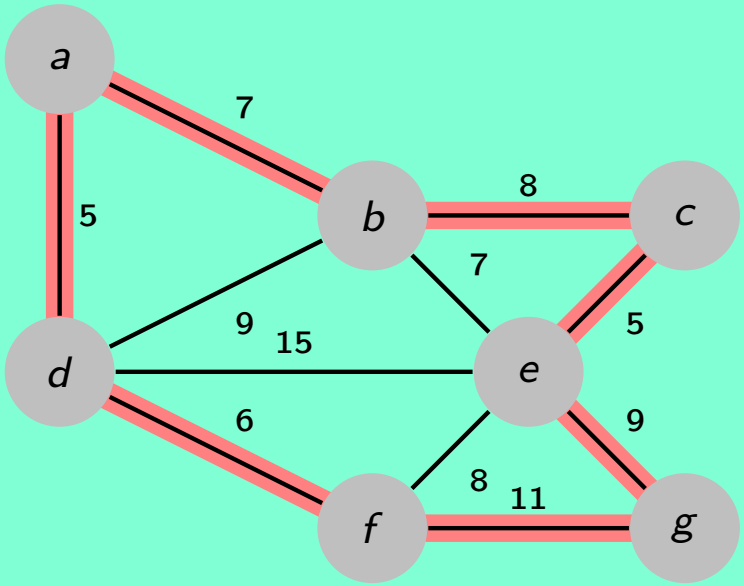

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What's a good NP-certificate for SAT and how to verify it?