Algorithms and Complexity (AC)

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(Based on slides by Gerhard Woeginger and Jesper Nederlof)

Landelijk Netwerk Mathematische Besliskunde

LNMB, Sep–Nov 2021

Note

Shorter lecture today.
Remainder of the lecture (NP-completeness proofs) is recorded as videos, see email (and coming on website soon)

Please type in the chat '[your name] is here' for the attendance list.

Definition: Complexity class NP

A decision problem X lies in the complexity class NP, if

- if it can be solved in polynomial time on a non-deterministic Turing machine (original, formal definition)
- (or, alternatively:) if it is solved by a *non-deterministic* algorithm with polynomial time complexity.
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Remark: A decision problem X lies in the complexity class co-NP, if **the NO-instances** of X possess certificates of polynomial length that can be verified in polynomial time. More on this in lecture 3.

Example for problem in NP: Subset Sum

Decision problem Subset Sum (SS)

Instance: positive integers $a_1, \ldots, a_n$; a bound b

Question: does there exist an index set $I \subseteq \{1, \ldots, n\}$ with $\sum_{i \in I} a_i = b$?

Example: $(a_1, \ldots, a_{12}) = (1, \ldots, 12), b = 50$. Yes or no instance?

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Yes: $1 + 2 + 3 + 4 + 6 + 7 + 8 + 9 + 10 = 50$.

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What's a good NP-certificate for SS?

Example for problem in NP: Exact cover

Decision problem Exact cover (Ex-Cov)

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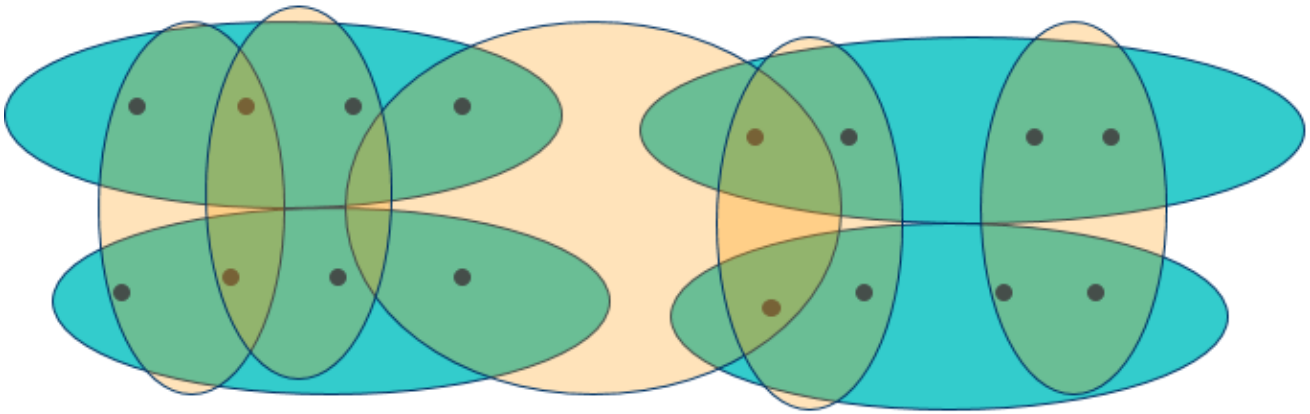

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Example for problem in NP: Integer programming

Integer linear programming (ILP)- Decision version

Instance: an integer matrix A; an integer vector b

Question: does there exist an integer vector x with $Ax \leq b$?

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P versus NP

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  P stands for Polynomial Time
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Answer YES:
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Answer NO:
- that's what most people expect

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is straightforward (although not always easy):

$\rightarrow$ Find an algorithm that runs in time $O(n \log n)$ / $O(n^3)$ / ... / polynomial time / non-deterministic polynomial time.

How do we prove that a problem can**not** be solved in a certain time?

"I can't find an efficient algorithm, because no such algorithm is possible!"

Proving lower bounds (short note)

Example: 'Find maximum element from unsorted list'

Input: A list of numbers $m_1, m_2, \ldots m_n$, a number M.
Question: Is there an element $\geq M$ in the list.

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Note: Most problems need time $\Omega(n)$ to be solved.
Can you think of one that does not?

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3-Satisfiability (3-SAT)

Instance:

a set of logical variables $X := \{x_1, \ldots, x_n\}$ and a set of clauses C of three literals over X

Question: does there exist a truth assignment for X that simultaneously satisfies all clauses in C?

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- Best found randomized algorithm similar ($O(1.306995^n)$?)
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Lower bounds are difficult to prove!

Lower bounds on problem complexity tend to be rare / weak / difficult to prove.

$\rightarrow$ We look at a different approach.

"I can't find an efficient algorithm, but neither can all these famous people."

Reductions

For a decision problem X, we denote by YES(X) the instances of that problem for which the answer is 'yes'.

Definition

For two decision problems X and Y, we say that X (polynomially) **reduces** to Y (and we write $X \leq_p Y$)
if there exists a polynomial time transformation f that translates instance of X into instances of Y with $I \in$YES(X) $\iff$ $f(I) \in$YES(Y).

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Intuition:

- X can be modelled as a special case of Y
- the 'computational hardness' of X is upper bounded by Y's
- If Y is easy, then also X is easy
- If X is difficult, then also Y is difficult

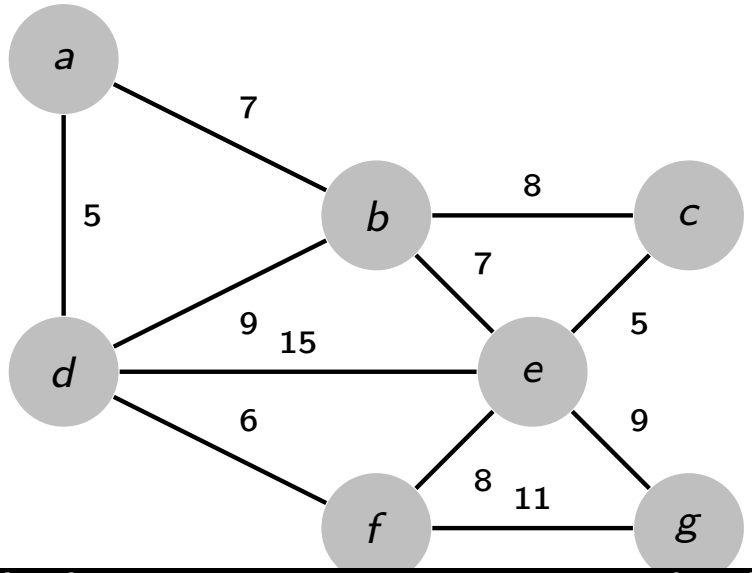
Hamiltonian cycle / TSP

Hamiltonian cycle (HC)

Instance: an undirected graph $G = (V, E)$
Question: does G contain a Hamiltonian cycle?
(a simple cycle that visits every vertex exactly once)

Travelling Salesman Problem (TSP)

Instance: cities $1, \ldots, n$; distances $d(i, j)$; a bound B
Question: does there exist a roundtrip of length at most B?



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Proof: next slide .

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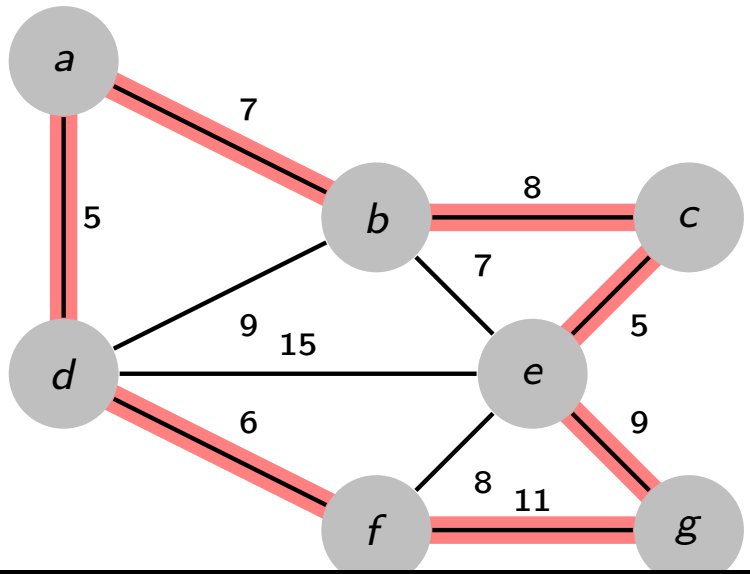

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- This transformation f can be done in polynomial time
- If I is a yes-instance of HC $\Rightarrow$ $f(I)$ is a yes-instance of TSP
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Furthermore, $d(c_{(i)}, c_{(i+1)}) = 1$ for all $i = 1, \ldots, n-1$ and $d(c_{(n)}, c_{(1)}) = 1$, because otherwise R would be longer than n
$\Rightarrow$ The sequence of nodes in G $C := (v_{(1)}, v_{(2)}, v_{(3)}, \ldots, v_{(n)}, v_{(1)})$ is a cycle, because $\{c_{(i)}, c_{(i+1)}\}$ for all $i = 1, \ldots, n-1$ and $\{c_{(n)}, c_{(1)}\}$ are contained in E

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Furthermore, $d(c_{(i)}, c_{(i+1)}) = 1$ for all $i = 1, \ldots, n-1$ and $d(c_{(n)}, c_{(1)}) = 1$, because otherwise R would be longer than n
$\Rightarrow$ The sequence of nodes in G $C := (v_{(1)}, v_{(2)}, v_{(3)}, \ldots, v_{(n)}, v_{(1)})$ is a cycle, because $\{c_{(i)}, c_{(i+1)}\}$ for all $i = 1, \ldots, n-1$ and $\{c_{(n)}, c_{(1)}\}$ are contained in E
$\Rightarrow$ I is a yes-instance of HC ✓

Proof: HC polynomially reduces to TSP

Let I be an instance of HP, consisting of undirected graph $G = (V, E)$.
We want to tranform I into an instance $f(I)$ of TSP.

We define the polynomial transformation f of instance I of HC to instance $f(I)$ of TSP:

- cities: introduce one city c_i for each $v_i \in V$
- distances: set $d(v, w) := \begin{cases} 1 & \text{if } e = \{v, w\} \in E \\ n+1 & \text{otherwise} \end{cases}$
- bound: $B := n$

This transformation can be done in polynomial time. ✓
If I is a yes-instance of HC $\Rightarrow$ $f(I)$ is a yes-instance of TSP ✓

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Clique

Instance: a graph $G = (V, E)$; an integer k
Question: does G contain a clique of size (at least) k?

Theorem

SAT $\leq_p$ CLIQUE.

Proof:

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Let t be a satisfying truth assignment. For each i, denote by $l(i)$ the first literal in clause c_i that is true. Then $V' := \{(l(i), i) : i = 1, \ldots, m\}$ is a clique, because any two nodes $(l(i), i)$, $(l(i'), i')$ are connected by an edge, as $i' \neq i$ and $l(i') \neq \neg l(i)$, as $l(i)$ and $\neg l(i)$ cannot be simultaneously true. ✓

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Then due to the definition of the edges in G, the truth assignment is conflict-free (so far).
For the remaining variables x we define $t(x) = \text{true}$.
As the size of the clique V' is m, and we now that no two nodes (l, i) and (l', i) are incident in G, we know that the clique contains a node (l, i) for each $i = 1, \ldots, m$. So each clause c_i has at least one literal l with $t(l) = \text{true}$. ✓

Lemma

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$X \leq_p Y$ and $Y \leq_p Z$ implies $X \leq_p Z$

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Proof: by putting the two tranformations into series

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if **all** problems $Y \in NP$ can be reduced to it
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Intuition:

- NP-complete problems are the hardest problems in NP
- Recall: NP is huge and contains tons of important problems
- Some people consider NP-complete problems to be intractable.

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NP-hardness

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Why? Can reduce to X and then solve produced instance of X.

"I can't find an efficient algorithm, but neither can all these famous people."

Cook-Levin theorem (1971)

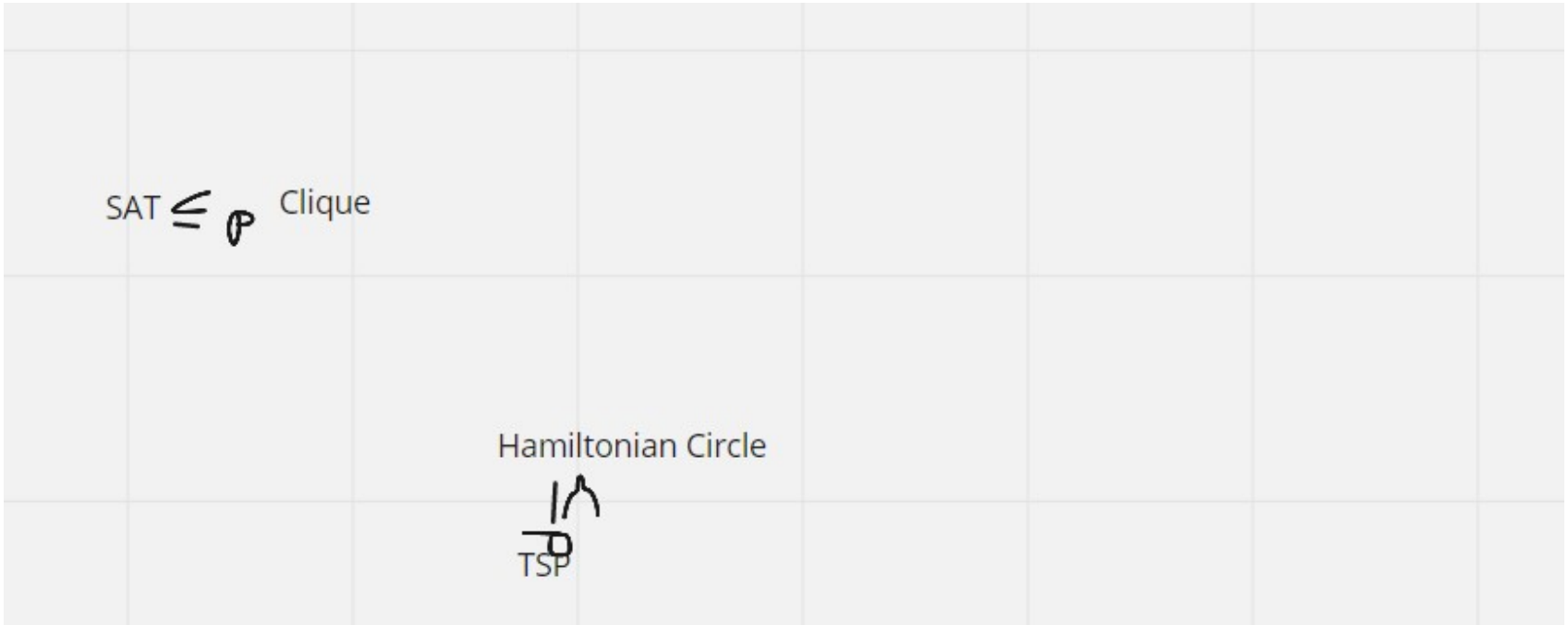
SAT is NP-complete.

see video for proof

- Stephen Cook (born 1939):
  American-Canadian computer scientist and mathematician
- Leonid Levin (born 1948):
  Russian computer scientist, discovered the result somewhat earlier

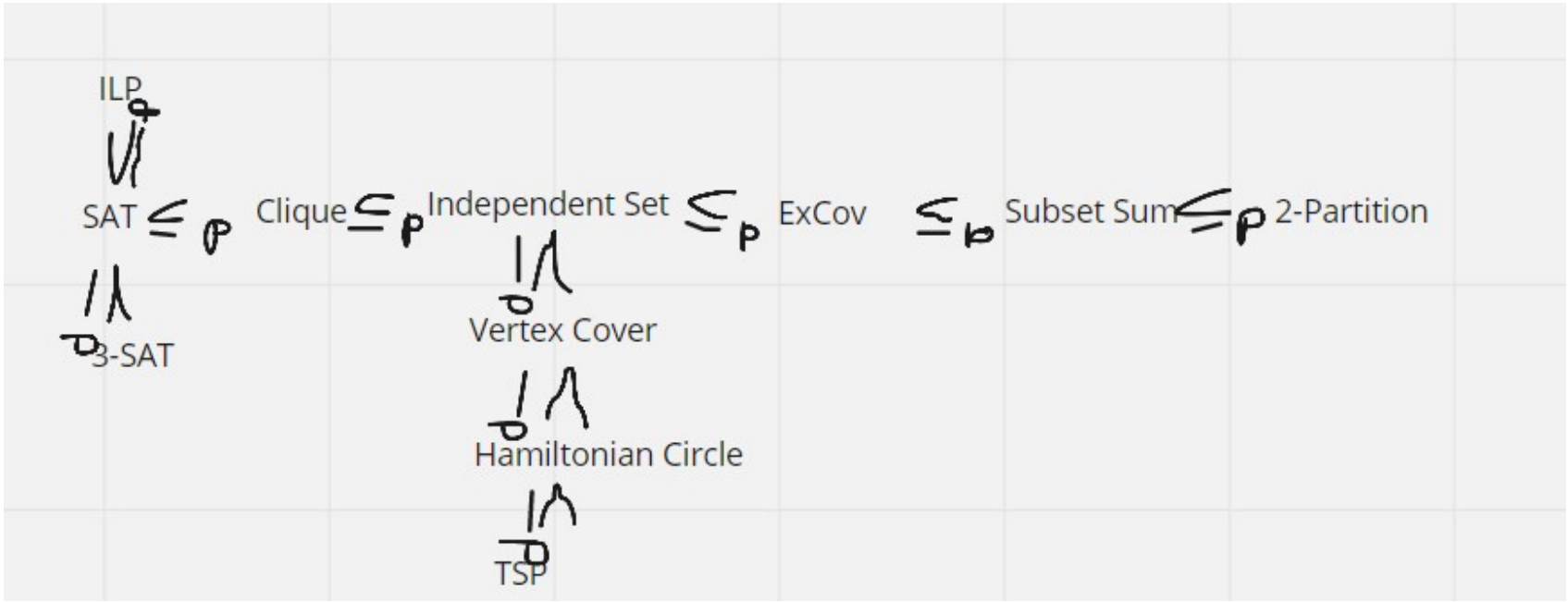
Videos

from here:



Videos

to here:



Recommended reading

Garey and Johnson. 'Algorithms and Complexity'
Lenstra and Rinnooy Kan. Computational complexity of discrete optimization problems.
Annals of Discrete Mathematics 4 (pp 121-140), 1979.
Electronic copy available on website

Cormen, Leiserson, Rivest and Stein 'Introduction to Algorithms':

- Chapter 1-3 (basics)
- Chapter 23 (minimum spanning trees)
- Chapter 34 (P, NP, NP-completeness, Cook-Levin theorem, reductions)