

Exercise set 2 Algorithms and Complexity 2021**Due 18 Oct 2021**

You may collaborate and submit answers in groups of two (or at most three). Good solutions are complete and concise. Please mail to `schmidt2@rsm.nl` or hand in a hardcopy on the day of the deadline.

3. An instance of the MAX BISECTION problem consists of an undirected graph with vertex set $V = \{1, 2, \dots, 2n\}$ and positive real edge weights $w(i, j)$ for $\{i, j\} \in E$. The goal is to partition V into two parts V_1 and V_2 of size n , so that the total weight of the edges between V_1 and V_2 is as large as possible. Consider the following ILP.

$$\begin{aligned} \max \quad & \sum_{\{i,j\} \in E} w(i, j) z_{i,j} \\ \text{s.t.} \quad & z_{i,j} \leq x_i + x_j \quad \text{for } \{i, j\} \in E \\ & z_{i,j} \leq 2 - x_i - x_j \quad \text{for } \{i, j\} \in E \\ & \sum_{i=1}^{2n} x_i = n \\ & x_i \in \{0, 1\} \quad \text{for } i \in V \\ & z_{i,j} \in \{0, 1\} \quad \text{for } \{i, j\} \in E \end{aligned}$$

- (a) Show that this ILP correctly models the MAX BISECTION problem.
 (b) Show that any optimal solution x and z of the ILP satisfies $z_{ij} = x_i + x_j - 2x_i x_j$.

Next, let us consider the LP relaxation of this ILP, where the integrality constraints $x_i \in \{0, 1\}$ and $z_{i,j} \in \{0, 1\}$ are relaxed to the continuous constraints $0 \leq x_i \leq 1$ and $0 \leq z_{i,j} \leq 1$. Furthermore we define the auxiliary value $F(x) := \sum_{\{i,j\} \in E} w(i, j) (x_i + x_j - 2x_i x_j)$.

- (c) Prove that any feasible solution x and z of the LP relaxation satisfies the inequality $F(x) \geq \frac{1}{2} \sum_{\{i,j\} \in E} w(i, j) z_{i,j}$.¹
 (d) Consider a feasible solution for the LP with the property that for each edge $\{i, j\} \in E$ either $x_i \geq \frac{1}{2} \geq x_j$ or $x_j \geq \frac{1}{2} \geq x_i$ which has at least two fractional variables, x_i and x_j . Argue that it is possible to increase one variable by $\epsilon > 0$ and to decrease the other one by the same ϵ , such that the solution stays feasible for the LP, the value of $F(x)$ does not decrease and one of the two variables becomes integer.
 (e) Use these arguments to design a polynomial time approximation algorithm for MAX BISECTION that yields at least 1/2 of the optimal objective value.
4. An instance of the MIN-RADIUS-FACILITY-LOCATION (MRFL) problem consists of an integer k , cities $\{1, \dots, n\}$, and integer distances $d(i, j)$ for every pair of cities i, j . The distances have the following properties for all cities x, y, z :

$$(a) \ d(x, x) = 0, \quad (b) \ d(x, y) = d(y, x), \quad (c) \ d(x, y) + d(y, z) \geq d(x, z).$$

If $C \subseteq \{1, \dots, n\}$ is a set of cities and j is a city, we denote $\text{dist}(C, j) := \min\{d(i, j) : i \in C\}$ for the minimum distance from j to a city in C (so $\text{dist}(C, j) = 0$ if $j \in C$). The goal in the MRFL problem is to find a set C of size at most k minimizing the radius $r(C) := \max_{1 \leq j \leq n} \text{dist}(C, j)$.

¹Hint: Use the AM-GM inequality (stating that, for non-negative a, b , $4ab \leq (a + b)^2$) to show that any feasible solution of the LP satisfies $\frac{1}{2} z_{i,j} \leq x_i + x_j - 2x_i x_j$ for every i and j .

- (a) Give a polynomial time algorithm that takes an instance of MRFL and a number t as input, and finds a set C of size at most k satisfying $r(C) \leq 2t$ if there exists a set C^* of size at most k with $r(C^*) \leq t$.
- (b) Use the algorithm from (a) to give a 2-approximation for MRFL. Your algorithm may run in pseudo-polynomial time.
- (c) **Bonus:**² Give a polynomial time 2-approximation for MRFL.

²You do not need to answer this question to get full points, but a fully correct answer to this question will give you full points for Question 4 plus a bonus point even when (a) and (b) are skipped.