Algorithms and Complexity (AC): Reductions

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(Based on slides by Gerhard Woeginger and Jesper Nederlof)

Landelijk Netwerk Mathematische Besliskunde

NP-hardness: 3-SAT

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Instance: a set of logical variables $X := \{x_1, \ldots, x_n\}$ and a set of clauses C of three literals over X

Question: does there exist a truth assignment for X that simultaneously satisfies all clauses in C?

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Proof: We show SAT $\leq_p$ 3-SAT.
Let $I = (X, C)$ an instance of SAT. We construct the following instance (X', C') of 3-SAT:

- For each clause c_j we construct a set of variables X_j and additional clauses C_j (with 3 literals each)
- $X' := X \cup \bigcup_{j=1}^{|C|} X_j$, $C' := \bigcup_{j=1}^{|C|} C_j$

Construction of X_j and C_j from clause c_j
1.) For $|c_j| = 1$ with $c_j = (l_j)$:

- $X_j = \{y_j^1, y_j^2\}$,
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- $C_j = \{(l_j \vee y_j^1 \vee y_j^2), (l_j \vee \neg y_j^1 \vee \neg y_j^2), (l_j \vee \neg y_j^1 \vee y_j^2), (l_j \vee y_j^1 \vee \neg y_j^2)\}$

2.) For $|c_j| = 2$ with $c_j = (l_j^1 \vee l_j^2)$:

- $X_j = \{y_j\}$, $C_j = \{(l_j^1 \vee l_j^2 \vee y_j), (l_j^1 \vee l_j^2 \vee \neg y_j)\}$

3.) For $|c_j| = 3$: $X_j = \emptyset$, $C_j = \{c_j\}$

4.) For $|c_j| > 3$ with $c_j = (l_j^1 \vee l_j^2 \vee \ldots \vee l_j^k)$

- $X_j = \{y_j^1, y_j^2, \ldots, y_j^{k-3}\}$
- $C_j = \{(l_j^1 \vee l_j^2 \vee y_j^1), (\neg y_j^1 \vee l_j^3 \vee y_j^2), (\neg y_j^2 \vee l_j^4 \vee y_j^3), \ldots, (\neg y_j^{k-3} \vee l_j^{k-1} \vee l_j^k)$

To show: 3. If there is a satisfying truth assignment for (X, C), then there is a satisfying truth assignment for $(X', C') = (X \cup \bigcup_{j=1}^{|C|} X_j, \bigcup_{j=1}^{|C|} C_j)$:

Define truth assignment t' as follows:

- $t'(x) := t(x)\ \forall x \in X$
- $t'(x) := \text{true}\ \forall x \in X_j$ with $|c_j| \in \{1, 2\}$
- For c_j with $|c_j| > 3$ let l_j^i be the first literal with $t(l_j^i) = \text{true}$: If $i \in \{1, 2\}$ we set $t'(y_j^i) := \text{false}\ \forall i$
  If $i \in \{k-1, k\}$ we set $t'(y_j^i) := \text{true}\ \forall i < k-3$
  If $i \in \{3, 4, \ldots, k-2\}$, set $t'(y_j^l) := \text{true}\ \forall\ l \leq i-2$ & $t'(y_j^l) = \text{true}\ \forall\ l \geq i-2$

Then all clauses in C' are satisfied.

Construction of X_j and C_j from clause c_j

1.) For $|c_j| = 1$ with $c_j = (l_j)$:

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2.) For $|c_j| = 2$ with $c_j = (l_j^1 \vee l_j^2)$:

- $X_j = \{y_j\}$, $C_j = \{(l_j^1 \vee l_j^2 \vee y_j), (l_j^1 \vee l_j^2 \vee \neg y_j)\}$

3.) For $|c_j| = 3$: $X_j = \emptyset$, $C_j = \{c_j\}$

4.) For $|c_j| > 3$ with $c_j = (l_j^1 \vee l_j^2 \vee \ldots \vee l_j^k)$

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Then all clauses in C' are satisfied. ✓

NP-hardness: Integer programming

Integer programming (ILP) - Decision version

Instance: an integer matrix A; an integer vector b
Question: does there exist an integer vector y with $Ay \leq b$?

Theorem

ILP is NP-hard (and NP-complete).

Proof:

NP-hardness: Integer programming

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Polynomial transformation:
Let (X, C) be an instance of SAT.
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To show:
This transformation is polynomial ✓

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encode $y_j \in \{0, 1\}$ as $0 \leq y_j \leq 1$.
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There is a satisfying truth assignment for $(X, C) \Leftrightarrow$ there is a vector y fulfilling $Ay \leq b$

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To show: There is a satisfying truth assignment for $(X, C) \Rightarrow$ there is a vector y fulfilling $Ay \leq b$

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Let t be a satisfying truth assignment for (X, C).

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To show: There is a satisfying truth assignment for $(X, C) \Rightarrow$ there is a vector y fulfilling $Ay \leq b$

Let t be a satisfying truth assignment for (X, C). We set $y_j := \begin{cases} 1 & \text{if } t(x_j) = \text{true} \\ 0 & \text{otherwise} \end{cases}$

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Then for each $c_i \in C$ we have

$$(a_{i\cdot}) \cdot y = \sum_{j=\mathbf{1}}^{n} a_{ij} y_j = \underbrace{\sum_{x_j \text{ occurs in } c_i} -y_j}_{\leq \mathbf{0}} + \underbrace{\sum_{\neg x_j \text{ occurs in } c_i} y_j}_{\leq |\{j : \neg x_j \text{ occurs in } c_i\}|} \leq |\{j : \neg x_j \text{ occurs in } c_i\}| - \mathbf{1}$$

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Let y be a vector with $Ay \leq b$.
Set $t(x_j) = \begin{cases} \text{true} & \text{if } y_j = 1 \\ \text{false} & \text{otherwise.} \end{cases}$

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Then for $c_i = l_1 \vee l_2 \vee \ldots \vee l_k$ we have

$$\sum_{x_j \text{ occurs in } c_i} -y_j + \sum_{\neg x_j \text{ occurs in } c_i} y_j |\{j : \neg x_j \text{ occurs in } c_i\}| - 1$$

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$\Rightarrow$ There is a j with x_j in c_i and $y_i = 1$, or there is a j with $\neg x_j$ in c_i and y_j=0

Polynomial transformation:
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Then for $c_i = l_1 \vee l_2 \vee \ldots \vee l_k$ we have

$$\sum_{x_j \text{ occurs in } c_i} -y_j + \sum_{\neg x_j \text{ occurs in } c_i} y_j |\{j : \neg x_j \text{ occurs in } c_i\}| - 1$$

$\Rightarrow$ There is a j with x_j in c_i and $y_i = 1$, or there is a j with $\neg x_j$ in c_i and y_j=0
$\Rightarrow$ c_i is satisfied ✓

NP-hardness: Integer programming

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Question: does there exist an integer vector y with $Ay \leq b$?

Theorem

SAT $\leq_p$ ILP, and therefore ILP is NP-hard (and NP-complete).

Consequence: Every problem in NP can be modelled as an ILP.

NP-hardness: Clique

Clique

Instance: a graph $G = (V, E)$; an integer k
Question: does G contain a clique of size (at least) k?

Theorem

CLIQUE is NP-hard (and NP-complete).

Proof: SAT is NP-hard and SAT $\leq_p$ CLIQUE.

NP-hardness: Independent set

Independent set (IS)

Instance: a graph $G = (V, E)$; an integer k
Question: does G contain an independent set of size (at least) k?
(a set of vertices that does not span any edge)

Theorem

IS is NP-hard (and NP-complete).

Proof:

NP-hardness: Independent set

Independent set (IS)

Instance: a graph $G = (V, E)$; an integer k
Question: does G contain an independent set of size (at least) k?
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Theorem

IS is NP-hard (and NP-complete).

Proof: By reduction from CLIQUE:

NP-hardness: Independent set

Independent set (IS)

Instance: a graph $G = (V, E)$; an integer k
Question: does G contain an independent set of size (at least) k?
(a set of vertices that does not span any edge)

Theorem

IS is NP-hard (and NP-complete).

Proof: By reduction from CLIQUE:
Given an instance $(G = (V, E), k)$ of clique, construct the following instance of IS:
$V' := V$, $E' := \{\{i, j\} : i \neq j \in V, \{i, j\} \notin E\}$, $k' := k$.

NP-hardness: Independent set

Independent set (IS)

Instance: a graph $G = (V, E)$; an integer k
Question: does G contain an independent set of size (at least) k?
(a set of vertices that does not span any edge)

Theorem

IS is NP-hard (and NP-complete).

Proof: By reduction from CLIQUE:
Given an instance $(G = (V, E), k)$ of clique, construct the following instance of IS:
$V' := V$, $E' := \{\{i, j\} : i \neq j \in V, \{i, j\} \notin E\}$, $k' := k$.
Show:
$X \subset V$ is a clique in $G \Leftrightarrow X$ is an independent set in G'

NP-hardness: Exact cover

Exact cover (Ex-Cov)

Instance: a ground set X; subsets $S_1, \ldots, S_m$ of X

Question: do there exist some subsets S_i that form a partition of X?

Theorem

(Ex-Cov) is NP-hard (and NP-complete).

Proof:

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Proof: by reduction from IS.
Let (G, k) with $G = (V, E)$ be an instance of IS.
Define an instance of (Ex-Cov) as follows: $X := E \cup \{1, \ldots, k\}$
and subsets
$S_{ih} := \{\{i, j\} : \{i, j\} \in E\} \cup \{h\}$ for $i \in V$, $h = 1, \ldots k$
$S_{\{i,j\}} := \{\{i, j\}\}$ for $\{i, j\} \in E$

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$S_{\{i,j\}} := \{\{i, j\}\}$ for $\{i, j\} \in E$
Show:
Polynomial transformation ✓
If S is a solution to (Ex-Cov), $i : S_{ih} \in S$ is an independent set of size k.
If $X = \{x_1, x_2, \ldots, x_k\} \subset V$ is an independent set, $\bigcup_{j=1}^{k} S_{x_j j} \cup \{\{i, j\} : i, j \notin X\}$ is a solution to (Ex-Cov).

NP-hardness: Exact cover

Let (G, k) with $G = (V, E)$ be an instance of IS.
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Show:If S is a solution to (Ex-Cov), $i : S_{ih} \in S$ is an independent set of size k:

For each $h = 1, \ldots, k$, there needs to be a $i(h)$ such that $S_{i(h)h)}$ is in the the partition.

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For any two $h \neq h'$, $i(h) \neq i(h')$ and vertices $i(h)$ and $i(h')$ are not adjacent.

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For any two $h \neq h'$, $i(h) \neq i(h')$ and vertices $i(h)$ and $i(h')$ are not adjacent.
$\Rightarrow \{i(h) : h = 1, \ldots, k\}$ is an independent set of size k in G. ✓

NP-hardness: Exact cover

Let (G, k) with $G = (V, E)$ be an instance of IS.
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Show: If $X = \{x_1, x_2, \ldots, x_k\} \subset V$ is an independent set,
$\bigcup_{j=1}^{k} S_{x_j j} \cup \{\{i, j\} : i, j \notin X\}$ is a solution to (Ex-Cov).

NP-hardness: Subset Sum

Subset Sum (SS)

Instance: positive integers $a_1, \ldots, a_n$; a bound b

Question: does there exist an index set $J \subseteq \{1, \ldots, n\}$ with $\sum_{j \in J} a_j = b$?

Theorem

SS is NP-hard (and NP-complete).

Proof:

NP-hardness: Subset Sum

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Proof: by reduction from Ex-Cov.

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Theorem

SS is NP-hard (and NP-complete).

Proof: by reduction from Ex-Cov.
Let $(X = \{x_1, \ldots, x_m\}, \{S_1, \ldots, S_n\})$ be an instance of Ex-Cov.
Define numbers a_j as $a_j := \sum_{i=1}^{m} c_{ij} \cdot d_i$ with $c_{ij} = 1$ if $x_i \in S_j$ and $d_i = (n+1)^{i-1}$.
Set $b := \sum_{i=1}^{m} (n+1)^{i-1}$.

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Set $b := \sum_{i=1}^{m} (n+1)^{i-1}$.

Show: 1.) This is a polynomial transformation.
2. & 3.) J is the index set of a solution to Ex-Cov $\Leftrightarrow$ J is the index set of a solution to SS.

Reduction of Ex-Cov to SS

Let $(X = \{x_1, \ldots, x_m\}, \{S_1, \ldots, S_n\})$ be an instance of Ex-Cov.
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Representation of Ex-Cov instance in table:

	x_1	x_2	$\ldots$	x_m	
S_1	1	0	$\ldots$	1	
S_2	1	1	$\ldots$	1	
$\ldots$					
S_n	0	1	$\ldots$	0	

✓

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We have subsets $\{S_i :\in I\}$ that form a partition (thus: a YES-instance)

$\Leftrightarrow$ we can choose subset of rows (with indices in index set I) such that there is exactly one entry '1' in each column.

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Representation of Ex-Cov instance in table:

	x_1	x_2	$\ldots$	x_m	
S_1	1	0	$\ldots$	1	$a_1 = 1 \cdot 1 + 0 \cdot (n+1) + \ldots + 1 \cdot (n+1)^{m-1}$
S_2	1	1	$\ldots$	1	$a_2 = 1 \cdot 1 + 1 \cdot (n+1) + \ldots + 1 \cdot (n+1)^{m-1}$
$\ldots$					
S_n	0	1	$\ldots$	0	$a_n = 0 \cdot 1 + 1 \cdot (n+1) + \ldots + 0 \cdot (n+1)^{m-1}$
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	x_1	x_2	$\ldots$	x_m	
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$\Leftrightarrow$ we can choose a subset of numbers (with indices in index set I) such that $\sum_{i \in I} a_i = \sum_{i=1}^{m} (n+1)^{i-1} = b$
✓

NP-hardness: 2-Partition

2-PARTITION

Instance: positive integers $a_1, \ldots, a_n$ with $\sum_{i=1}^{n} a_i = 2A$.

Question: does there exist an index set $I \subseteq \{1, \ldots, n\}$ with $\sum_{i \in I} a_i = A$?

Theorem

2-PARTITION is NP-hard (and thus NP-complete).

Proof:

NP-hardness: 2-Partition

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Given an instance $(\{a_1, \ldots, a_n\}, b)$ of SS we construct an instance of 2PARTITION:

NP-hardness: 2-Partition

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Given an instance $(\{a_1, \ldots, a_n\}, b)$ of SS we construct an instance of 2PARTITION:
we define $S := \sum_{i=1}^{n} a_i$
and consider the instance $\{a_1, \ldots, a_n, \underbrace{S + b}_{=:a_{n+1}}, \underbrace{2S - b}_{=:a_{n+2}}\}$.

NP-hardness: 2-Partition

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Note that $\sum_{i=1}^{n+2} a_i = a_1 + a_2 + \ldots, a_n + S + b + 2S - b = 4S$.

NP-hardness: 2-Partition

2-PARTITION

Instance: positive integers $a_1, \ldots, a_n$ with $\sum_{i=1}^{n} a_i = 2A$.

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The question (of the built 2-PARTITION instance) is thus: Is there a an index set $I' \subset \{1, 2, \ldots, n+2\}$ with $\sum_{i \in I'} a_i = 2S$?

NP-hardness: 2-Partition

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The question (of the built 2-PARTITION instance) is thus: Is there a an index set $I' \subset \{1, 2, \ldots, n+2\}$ with $\sum_{i \in I'} a_i = 2S$?

We show: There is an index set $I \subset \{1, 2, \ldots, n\}$ with $\sum_{i \in I} a_i = b$
$\Leftrightarrow$ There is an index set $I' \subset \{1, 2, \ldots, n+2\}$ with $\sum_{i \in I} a_i = 2S$

NP-hardness: 2-Partition

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$\Rightarrow$
Assume that there is an index set $I \subset \{a_1, a_2, \ldots, a_n\}$ such that $\sum_{i \in I} a_i = b$.

NP-hardness: 2-Partition

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Assume that there is an index set $I \subset \{a_1, a_2, \ldots, a_n\}$ such that $\sum_{i \in I} a_i = b$.
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Then for $I' := I \cup \{n+2\}$ we have $\sum_{i \in I'} a_i = \sum_{i \in I} a_i + 2S - b = b + 2S - b = 2S$.
✓

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Assume that there is an index set $I' \subset \{1, 2, \ldots, n+2\}$ with $\sum_{i \in I} a_i = 2S$:
This index set contains either $n+1$ or $n+2$, but not both, because
$a_{n+1} + a_{n+2} = S + b + 2S - b = 3S > 2S$. Let's (wlog) assume that $n+1 \in I'$.

NP-hardness: 2-Partition

We show: There is an index set $I \subset \{1, 2, \ldots, n\}$ with $\sum_{i \in I} a_i = b$
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Then we define $I := I' \setminus \{n+1\}$.

NP-hardness: 2-Partition

We show: There is an index set $I \subset \{1, 2, \ldots, n\}$ with $\sum_{i \in I} a_i = b$
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$a_{n+1} + a_{n+2} = S + b + 2S - b = 3S > 2S$. Let's (wlog) assume that $n+1 \in I'$.
Then we define $I := I' \setminus \{n+1\}$. And have $\sum_{i \in I' \setminus \{n+1\}} a_i = 2S - (S - b) = b$ ✓

Vertex cover (VC)

Instance: a graph $G = (V, E)$; an integer k
Question: does G contain a vertex cover of size (at most) k?
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VC is NP-hard (and thus NP-complete).

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We consider the instance $(G = (V, E), n - l)$ of Vertex Cover.

V^1 is an independent set of size l in $G \Leftrightarrow V^2 := V \setminus V^1$ is a vertex cover of size $n - l \in G$

NP-hardness: Hamiltonian cycle / TSP

Hamiltonian cycle (HC)

Instance: a graph (V, E)
Question: does this graph contain a Hamiltonian cycle?

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HC is NP-complete.

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HC is NP-complete.

Proof: Easy to see: in NP.
To show NP-hard: reduction from VC.

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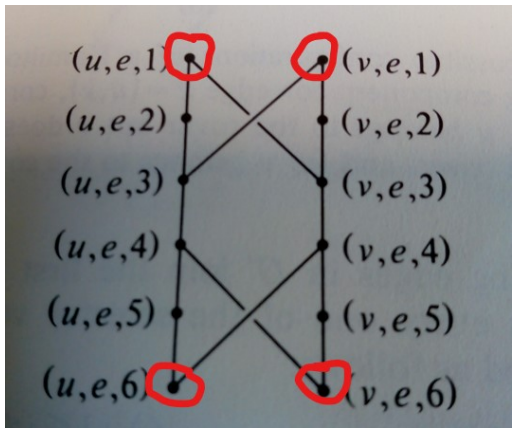

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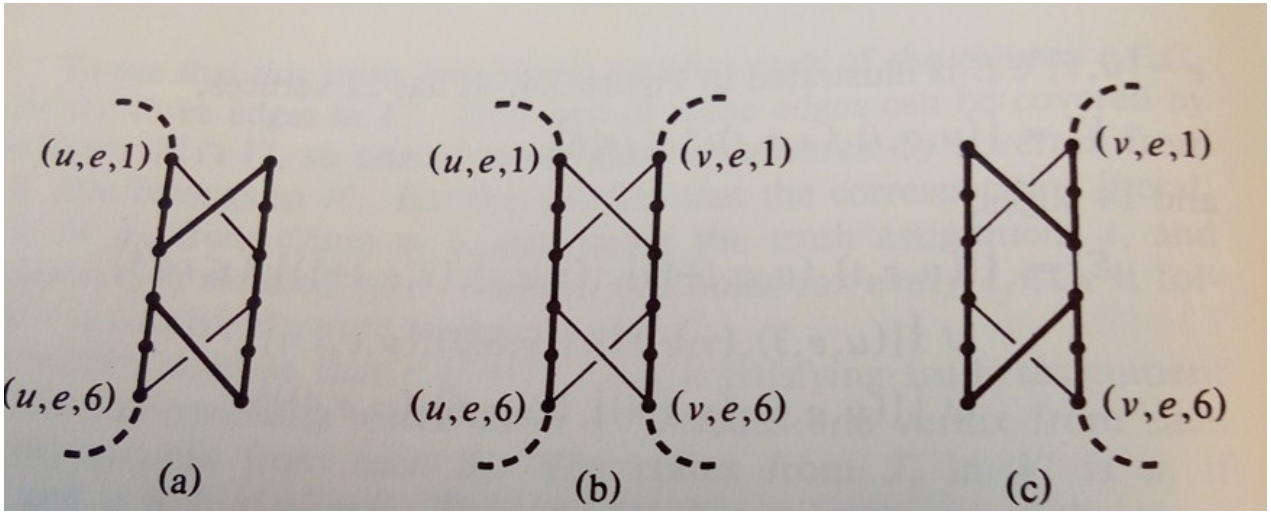

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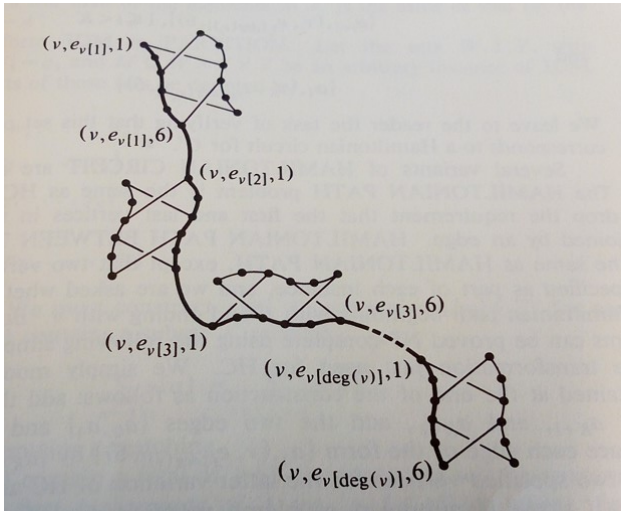

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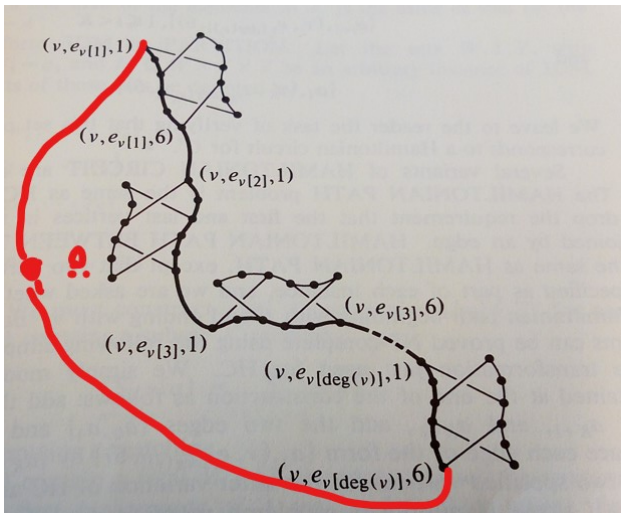

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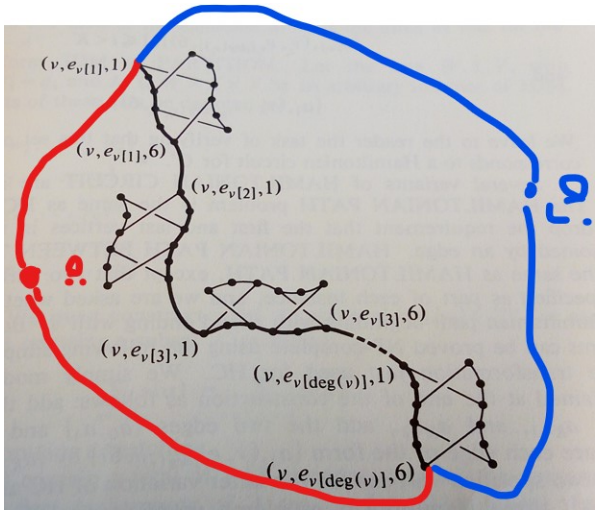

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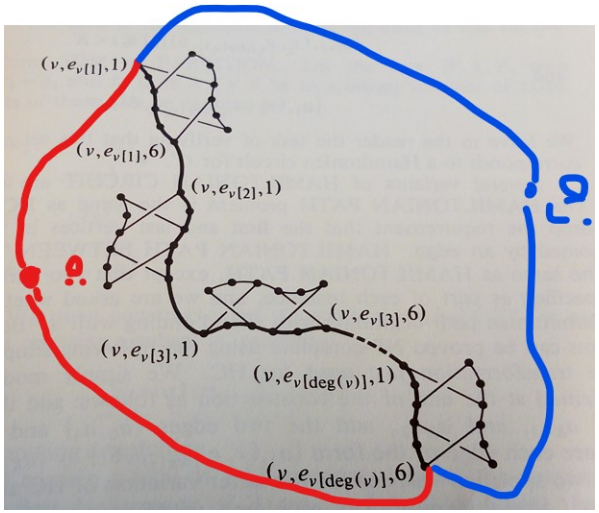
Transformation in polynomial time ✓



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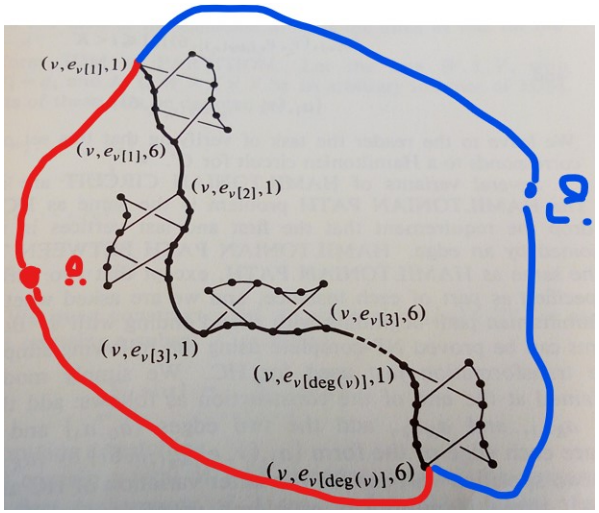
Show: G' has a Hamiltonian cycle $C \Rightarrow G$ has a vertex cover of size k:



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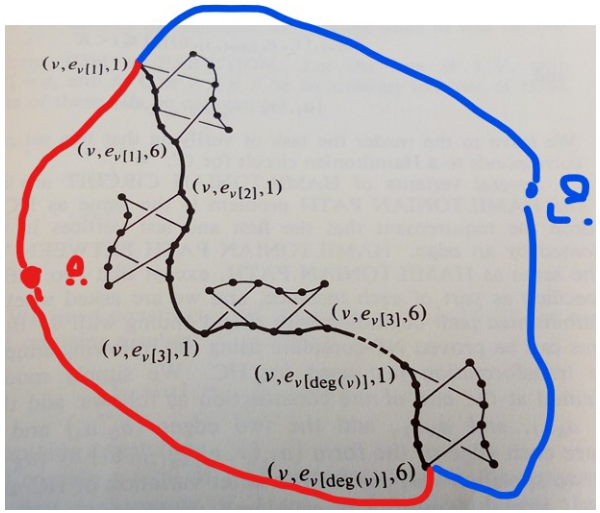

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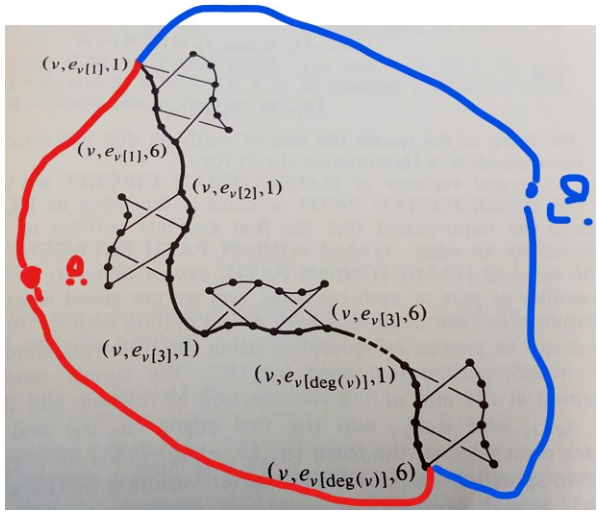

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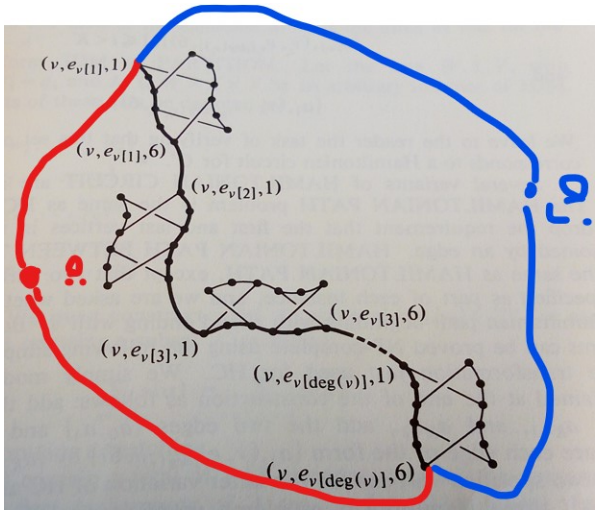

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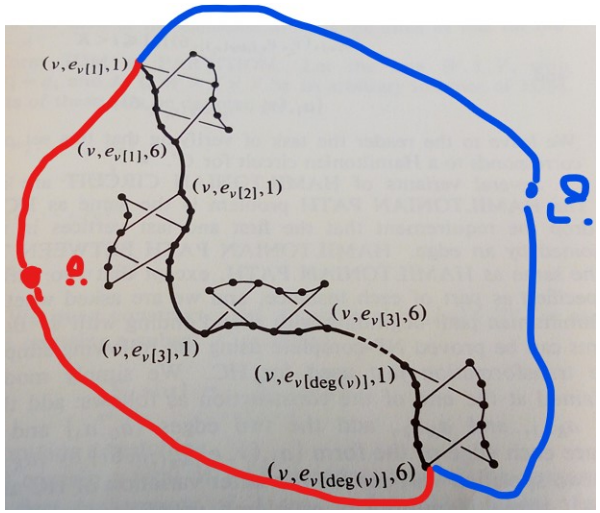

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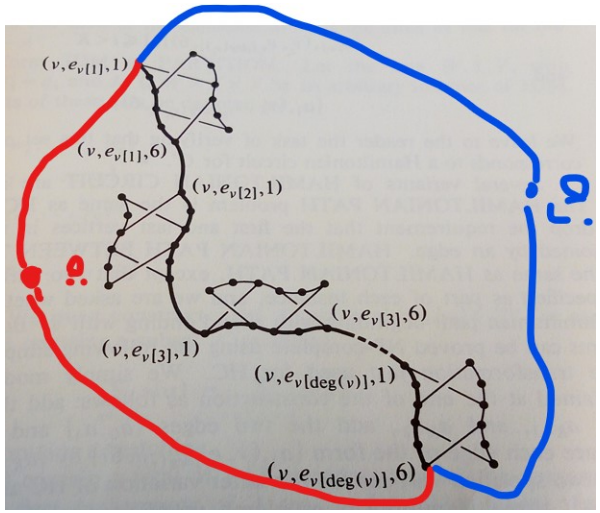
If there was an edge that was not covered in E, then its 'cover-testing component' would not be visited. ✓



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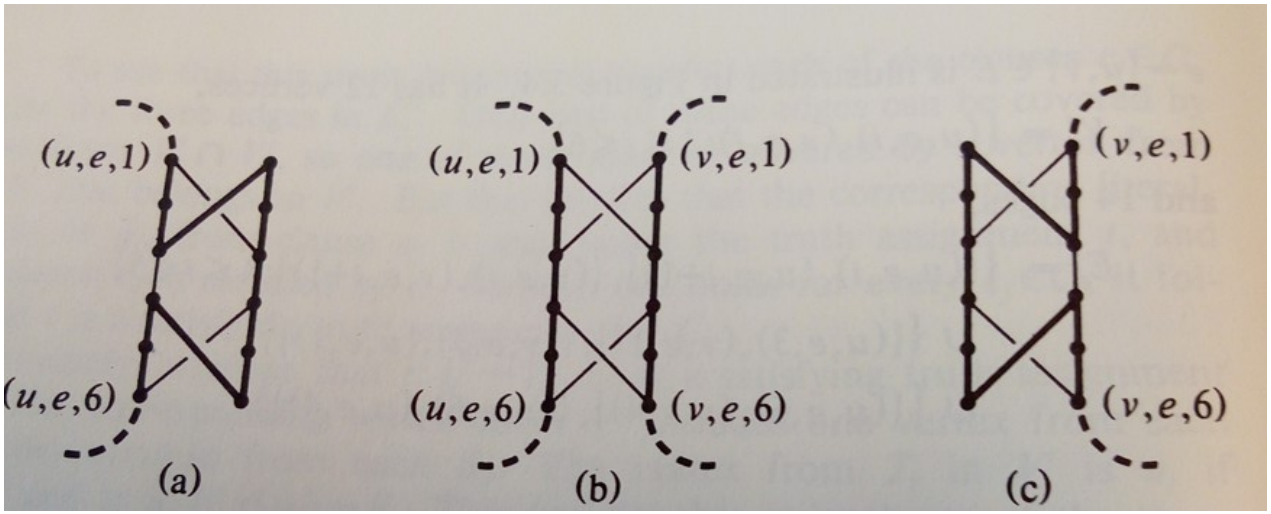

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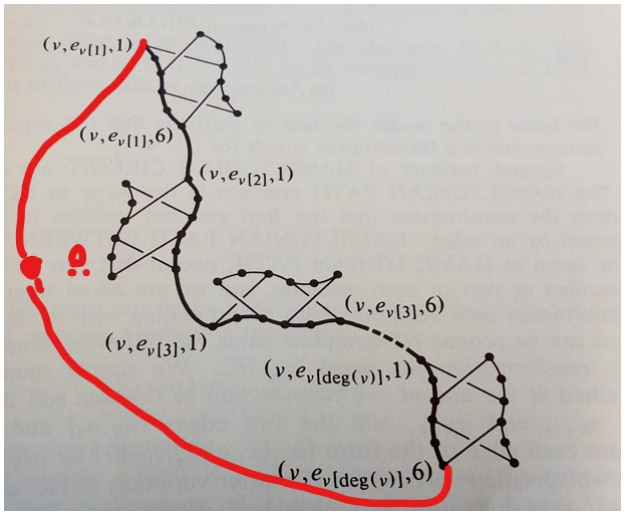

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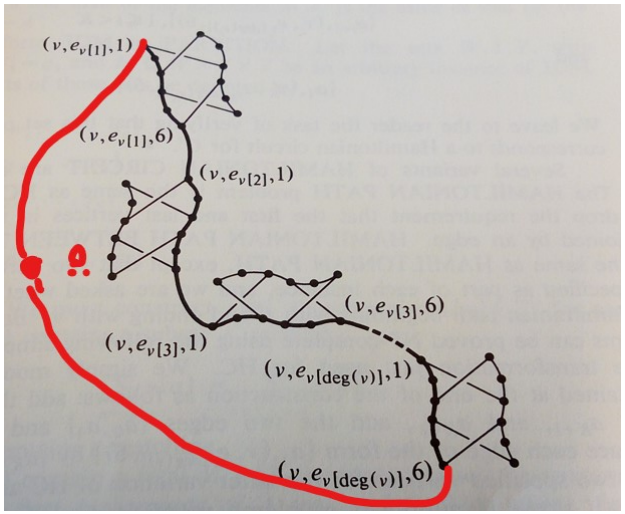

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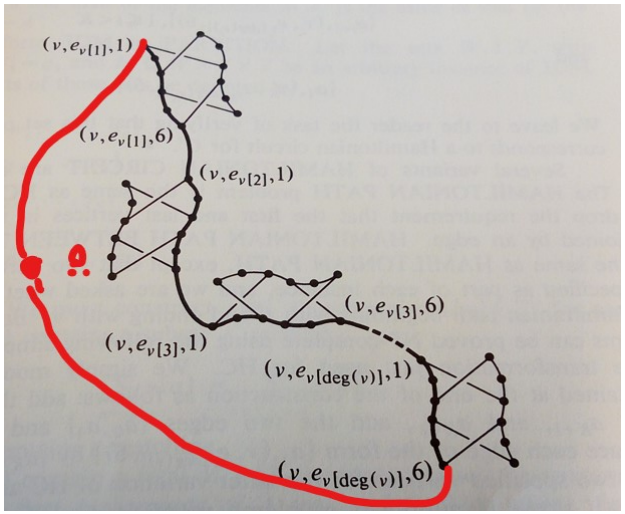
This forms a Hamiltonian cycle ✓



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Theorem

TSP is NP-complete.

Proof: already seen: in NP and TSP$\leq_p$ HC.